A finite volume method for numerical simulation of turbulence flows using a $k-\varepsilon$ model

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Abstract

We consider in this work an expanded finite volume numerical approximation of the turbulent $k-\varepsilon$ shallow water equations, on unstructured meshes, we use a simple discretization in wish only physical fluxes and averaged states are used in their formulations. To control the local diffusion in the scheme and also to preserve monotonicity, a parameter is introduced based on the sign matrix of the flux Jacobian. Numerical results are presented and compared with experimental data, for a backward-facing flow problem, to demonstrate and confirm its capability to provide accurate simulation of turbulent flows.

Keywords: *Shallow water equations; Turbulence model; Depth-averaged $k-\varepsilon$ model; Finite volume.*

1. Introduction

Two-dimensional depth averaged models have become very popular since the last decades, and the shallow water equations in depth-averaged form have been successfully applied to many engineering problems, and their application fields include a wide spectrum of phenomena other than water waves. For instance, the shallow water equations have applications in environmental and hydraulic engineering such as tidal flows in an estuary or coastal regions, rivers, reservoir, and open channel flows. Such practical flow problems are not trivial to simulate because the geometry can be complex, and the topography tends to be irregular. In addition, most of water free-surface flows encountered in engineering practice are turbulent characterized. This property makes direct numerical simulation of turbulent flows very difficult.

In this paper, we adopt the depth-averaged $k-\varepsilon$ model proposed in [1], which was the first depth-averaged two-equation eddy viscosity model, and it is still the most commonly used with the depth-averaged models when turbulent effects are accounted for in the computation. The depth-averaged $k-\varepsilon$ model constitutes an alternative for direct numerical simulation and large eddy simulation in industrial codes. Their advantage lies in the fact that the resolved quantities are assumed to be deterministic and therefore require no effective capture of random fluctuations, especially in the near-wall regions. As a direct consequence, the spatial discretization involved may be significantly more larges. More details concerning the depth-averaged models can be found in the references [2,3] and for research studies on modeling and numerical simulation of turbulent shallow water flows, we cite, for example, [4,5].

In the current work, we propose an enhanced finite volume method that incorporates the techniques of the methods proposed in [6]. Our main goal is to present a class of efficient numerical methods that can accurately solves the turbulent $k-\varepsilon$ shallow water equations. We, firstly, rearranged the turbulent $k-\varepsilon$ shallow water equations in a model forms a hyperbolic system of conservation laws with source terms. Then, the finite volume Non-Homogeneous Riemann Solver is used to solve this system. The method employs only physical fluxes and averaged states in their formulations. To control the local diffusion in the scheme and also to preserve monotonicity, a parameter is introduced based on the sign matrix of the flux Jacobian.

2. The turbulence shallow water equations

In conservation form, the two-dimensional non-linear shallow water equations are given by the depth-averaged continuity equation and the respective x- and y-depth-averaged momentum equations [7-8]:

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0 \qquad (1)$$

$$\frac{\partial (hu)}{\partial t}+\frac{\partial (hu^2+\frac{1}{2}gh^2)}{\partial x}+\frac{\partial (huv)}{\partial y}=-gh\frac{\partial z_b}{\partial x}-\frac{S_{fx}}{\rho}$$
$$+\frac{\partial}{\partial x}\left[(\upsilon+2\upsilon_t)\frac{\partial (hu)}{\partial x}\right]+\frac{\partial}{\partial y}\left[(\upsilon+\upsilon_t)\left(\frac{\partial (hu)}{\partial y}+\frac{\partial (hv)}{\partial x}\right)\right] \qquad (2)$$

$$\frac{\partial (hv)}{\partial t}+\frac{\partial (huv)}{\partial x}+\frac{\partial (hv^2+\frac{1}{2}gh^2)}{\partial y}=-gh\frac{\partial z_b}{\partial y}-\frac{S_{fy}}{\rho}$$
$$+\frac{\partial}{\partial x}\left[(\upsilon+\upsilon_t)\left(\frac{\partial (hu)}{\partial y}+\frac{\partial (hv)}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[(\upsilon+2\upsilon_t)\frac{\partial (hv)}{\partial y}\right] \qquad (3)$$

Where h is the total depth from the sea bed to the free surface, u and v are the depth-averaged velocity components in the Cartesian x and y directions, z_b is the bed elevation above a fixed horizontal datum, g the acceleration due to gravity, and S_{fy} and S_{fy} are the bed shear stress components, defined as

$$S_{fx} = \rho C_b u\sqrt{u^2+v^2},\ S_{fy} = \rho C_b v\sqrt{u^2+v^2} \quad (4)$$

where ρ is the water density and C_b is the bed friction coefficient, which may be estimated from $C_b = \frac{gn_M^2}{h^{1/3}}$, where n_M is the Manning coefficient, υ is the kinematic viscosity of water, and υ_t is a turbulent eddy viscosity, that quantify the energy dissipation due to the turbulent interactions among the particlesTo determine the turbulent eddy viscosity, the $k-\varepsilon$ model is used, in which k is the turbulence kinetic energy and ε is the dissipation rate per unit mass. The shallow water equations are obtained from depth integration, therefore it seems reasonable to use the same calcul of the $k-\varepsilon$ model [9]. Therefore, the turbulent eddy viscosity is calculated as: $\nu_t = C_\mu \frac{k^2}{\varepsilon}$ (5)

Where k and ε are given by the transport equation

$$\frac{\partial(h\zeta)}{\partial t} + \frac{\partial(hu\zeta)}{\partial x} + \frac{\partial(hv\zeta)}{\partial y} = \frac{\partial}{\partial x}\left[(\upsilon + \frac{\upsilon_t}{\sigma_\zeta})h\frac{\partial \zeta}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\upsilon + \frac{\upsilon_t}{\sigma_\zeta})h\frac{\partial \zeta}{\partial y}\right] + S_\zeta.\quad \zeta = (k,\varepsilon) \quad (6)$$

Wehre

$$\begin{cases} S_k = h[P_H + P_{kV} - \varepsilon] \\ S_\varepsilon = h\left[C_{\varepsilon 1}\frac{\varepsilon}{k}P_H + P_{\varepsilon V} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}\right] \end{cases} \quad (7)$$

With

$$P_H = 2\nu_t\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$P_{kV} = \frac{1}{\sqrt{C_f}}\frac{U^{*3}}{h} \quad \text{and} \quad P_{\varepsilon V} = 3,6\frac{C_{\varepsilon 2}\sqrt{C_\mu}}{C_f^{3/4}}\frac{U^{*4}}{h^2}$$

P_H is the production of k due to interactions of turbulent stresses with horizontal mean velocity gradients, and P_{kV} and $P_{\varepsilon V}$ are the productions of k and ε due to vertical velocity gradients and are related to the friction velocity U^*. The friction coefficient C_f can be obtained as:

$$U^* = \sqrt{C\mu\frac{k^2}{\varepsilon}|\partial U|)} \quad (8)$$

Finally, the values of the empirical constants considered in this study are.

$C_\mu = 0.09$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $\sigma_k = 1.2$ and $\sigma_\varepsilon = 1.3$.

3. SNRH finite volume method

For simplify, using matrix-vector notation, the two dimensional turbulent shallow water system can be written:

$$\mathbf{W}_t + \mathbf{F}_1(\mathbf{W})_x + \mathbf{F}_2(\mathbf{W})_y - \tilde{\mathbf{F}}_1(\mathbf{W})_x - \tilde{\mathbf{F}}_2(\mathbf{W})_y = \mathbf{S}(\mathbf{W}) \quad (9)$$

Where W is the vector of dependent variables, $\mathbf{F}_1, \mathbf{F}_2$ are the inviscid flux vectors, $\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2$ are diffusive flux vectors, S is the vector of source terms, and the subscripts x, y, and t denote partial differentiation.

In order to discretize system (9), the domain is meshed with a set of conforming triangular elements. A finite volume discretization of (9) yields.

$$\frac{(\mathbf{W}_i^{n+1} - \mathbf{W}_i^n)}{\Delta t}|V_i| + \sum_{j\in N(i)}\int_{\Gamma_{ij}} \mathbf{F}(\mathbf{W}^n, \hat{\mathbf{n}})\,d\Gamma - \sum_{j\in N(i)}\int_{\Gamma_{ij}} \tilde{\mathbf{F}}(\mathbf{W}^n, \hat{\mathbf{n}})\,d\Gamma = \int_{V_i} \mathbf{S}(\mathbf{W}^n)\,dV \quad (11)$$

Where $\mathbf{W}_i^n = \mathbf{W}(x_i, t_n)$ is the vector of conserved variables evaluated at time level $t_n = n\Delta$, n is the number of time steps, Δt is the time step, and $|V_i|$ is the area of cell V_i.

The construction of the numerical scheme is based on the hyperbolicity of the system and the self-similarity of the solution, it's consists of a predictor and corrector stages as:

$$\boldsymbol{W}_{ij}^n = \tfrac{1}{2}(\boldsymbol{W}_i + \boldsymbol{W}_j) - \tfrac{1}{2}\operatorname{sgn}\left[\boldsymbol{A}(F(\overline{\boldsymbol{W}_{ij}^n}, \boldsymbol{n}))\right](\boldsymbol{W}_j^n - \boldsymbol{W}_i^n) + \tfrac{1}{2}\left[\boldsymbol{A}(F(\overline{\boldsymbol{W}_{ij}^n}, \boldsymbol{n}))^{-1}\right]\boldsymbol{S}_{ij} \quad (12)$$

$$\frac{(\mathbf{W}_i^{n+1} - \mathbf{W}_i^n)}{\Delta t}|V_i| = \sum_{j\in N(i)}\int_{\Gamma_{ij}} \mathbf{F}(\mathbf{W}^n, \hat{\mathbf{n}})\,d\Gamma = \int_{V_i} \mathbf{S}(\mathbf{W}^n)\,dV \quad (13)$$

Where $A(F(\overline{W_{ij}^n}, \boldsymbol{n}))$ is the Jacobian matrix with respect to W_{ij}^n, and $\overline{W_{ij}^n}$ is approximated either by Roe's average state.

4. Results and discution

The results will be compared against data obtained by Fe et al. [10] who carried out an experimental and numerical study of recirculation in a water channel containing a sidewall expansion. The computational domain consists in a $0,5m \times 3,5m$ horizontal channel with an abrupt expansion located at $1m$ from the inlet, commonly known as backward facing step (show figure 2). The inflow discharge at the left boundary is enforced with $20.2l/s$ and imposed a water height of $24.2cm$ at the outflow boundary. the no-slip condition assumes that at a solid boundary, the fluid will have zero velocity is imposed for the rest. An unstructured triangular mesh are generated for the channel with consist of a 2000 nodes and 3691 elements (see figure 3).

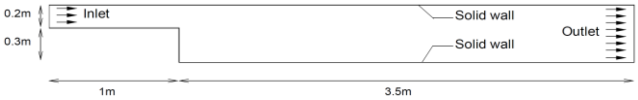

Figure 2: Flow in channel with a backward-facing step: Definition of problem domain

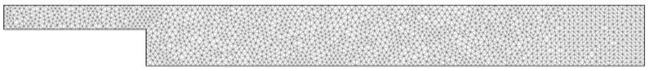

Figure 3: Flow in channel with a backward-facing step: Unstructured triangular mesh used in numerical model.

In Figure 4 and 5, we present a comparison between experimental and computational results; we illustrate the snapshot of the velocity and the kinetic energy along with the velocity fields. For clear presentation, we have also included streamlines within the presented results. From these results, we can see that the proposed method resolves accurately the flow structures, and the vortices seem to be localized in the correct place in the flow domain. For instance, the recirculation zone is in good agreement with the experimental measurements. For more comparisons, Figure 6 shows profiles of the velocity at $x = 1.53m$ within the measurement values. A good agreement between measured and predicted profiles has been obtained.

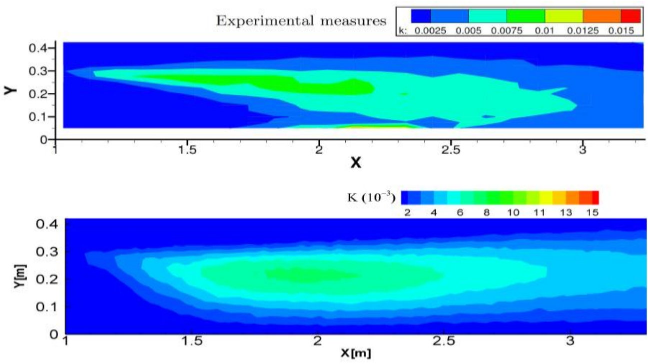

Figure 4: Flow in channel with a backward-facing step: Comparison of experimental (top) and numerical (bottom) results of kinetic energy k.

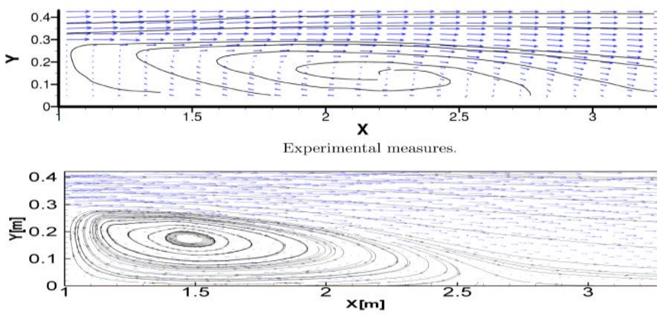

Figure 5: Flow in channel with a backward-facing step: Comparison of experimental (top) and numerical (bottom) results of velocity fields

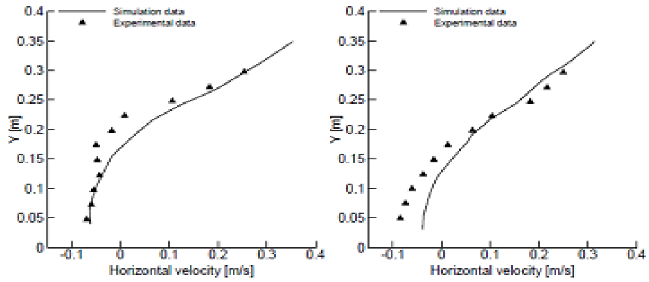

Figure 6: Flow in channel with a backward-facing step: a cross section at x=1.53m and x=2.03 of the velocity, with a comparison between experimental and numerical results.

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