Hydrodynamic stability of plane Poiseuille flow in Maxwell fluid with cross-flow

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Abstract

The linear stability analysis of plane Poiseuille flow in a Maxwell fluid in the presence of a uniform cross-flow is studied. The physical problem is reduced to a modified Orr-Sommerfeld equation with nonlinear eigenvalues and solved numerically using the Chebyshev spectral collocation method. Attention is focused on the combined effects of uniform cross-flow and the relaxation time of the fluid (Deborah number) on the flow stability. Results obtained in this framework show that, the cross-flow can either delay or advances the instability of this system, respectively for the case of a Newtonian fluid and for the case of a Maxwell one. In addition, the system is apt to lose its instability with the fluid's elasticity in short wavelength and to enhance this instability in the long wavelength regime.

Keywords: *hydrodynamic stability, Cross-flow, Maxwell fluid, Spectral Method, Nonlinear eigenvalues.*

1. Introduction

Since it was encountered in many industrial and technological applications, the stability of flows between two porous walls in the presence of cross-flow has received an upsurge interest during the last decay. One can cite for example: the biomedical industry, papermaking, filtration systems, environmental engineering and aeronautics. This configuration was initially carried out theoretically by Berman [1] in which a description of the laminar flow is discussed. In addition, Hains [2] and Sheppard [3] performed a linear stability analysis of the channel flow of Newtonian fluids, with an injection in the upper wall and a suction to the bottom one. In these studies, the authors have shown that a modest amount of uniform injection/suction of the same fluid produced a significant increase in critical Reynolds number. Indeed, they have defined two Reynolds numbers related respectively to, the maximum symmetric plane Poiseuille velocity and the cross-flow velocity. An extension of previous works [2, 3], Fransson and Alfredsson [4] have made corrections to the problems discussed in [2, 3]. In particular, they separated the effects of the velocity distribution from those of the magnitude of the velocity in the basic state, by using the maximal channel velocity of plane Poiseuille flow with the presence of a cross-flow as their characteristic velocity. Consequently, they proved the dependence of stability problem to the choice of the speed scale. In addition, they showed the stabilizing and destabilizing effect of transverse jet. Thus, they have mentioned that the cross-flow has stabilizing and destabilizing effects depending to the cross-flow rate. Recently, Lamine and Hifdi [5] showed that the cross-flow's sense has no effect on the stability of Poiseuille flow. In this paper, we extend the stability analysis of plane Poiseuille flow of a Newtonian fluid to that of a viscoelastic model. The considering model is that of a linear Maxwell fluid with a constant viscosity and a weak elasticity. This model can be used in certain industrial applications, especially in the analysis of small deformations of plastics. Furthermore, some particular fluid that often behave like a linear Maxwell model, for example the associative polymers such as hydrophobic ethoxylated urethane (HEUR) [6] and aqueous surfactant solutions containing threadlike micelles [7]. Recently, the effect of relaxation time of linear Maxwell model on the stability was examined out by Riahi et *al.* [8] who analyzed a linear stability of a pulsed flow in a linear Maxwell fluid in the Taylor-Couette system. Our investigation is focused to understand the combined effects of cross-flow and relaxation time in parallel shear flow. This paper is organized as follows. The studied configuration and the steady basic flow solution are defined in section 2. Section 3 is devoted to performing a linear stability analysis and to present the numerical method used to solve the stability problem. In section 3, pertinent results are discussed quantitatively.

2. Formulation and base-flow solution

Consider a plane channel flow of an incompressible fluid with the density, ρ , and the dynamic viscosity, μ. The channel is formed by two porous parallel plates separated by a fixed distance 2d. The upper and lower plates are located, respectively, at $y^* = +d$ and $y^* = -d$. A uniform cross-flow (injection/suction) of constant velocity, v_o, is imposed on the channel walls in the transverse direction, y^*. The injection at upper plate and suction at lower plate as shown in Fig.1. The mathematical equations modelling the physical problem in their dimensional forms (*) are, respectively, the continuity and Cauchy equations

$$div\mathbf{V}^* = 0 \tag{1}$$

$$\rho(\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^*\nabla\mathbf{V}^*) = -\nabla \mathrm{P}^* + \nabla \boldsymbol{\tau}^* \tag{2}$$

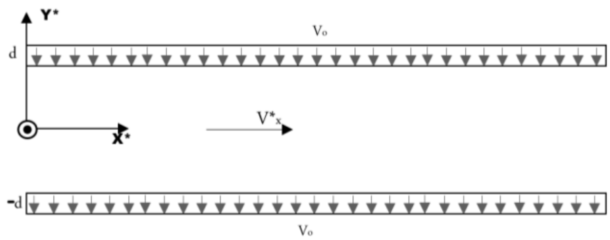

Figure 1 Sketch of the flow configuration.

where V^*, P^* and t^* are, respectively, the velocity the pressure and the time. $\boldsymbol{\tau}^*$ is the stress tensor, which can be written in a linear Maxwell's model fluid as follows

$$\boldsymbol{\tau}^* + \lambda \frac{\partial \boldsymbol{\tau}^*}{\partial t^*} = \mu \dot{\gamma} \quad (3)$$

with $$\dot{\gamma} = (\nabla V^* + \nabla^T V^*) \quad (4)$$

Here λ is the relaxation time and $\dot{\gamma}$ represents the rate of strain tensor. The boundary conditions at the walls ($y^* = \pm d$) are

$$V^*_{x^*} = 0 \;\; at \;\; y^* = \pm d \; ; V^*_{y^*} = -v_o \;\; at \;\; y^* = \pm d \quad (5)$$

Using reference variables $d, d/U^*_{\max}, U^*_{\max}$ and $\rho U^{*2}_{\max}$ for, respectively, length, time, velocity and pressure ($U^*_{\max}$ represents the maximum streamwise velocity), as follows

$$y = \frac{y^*}{d}, \; x = \frac{x^*}{d}, \; t = \frac{U^*_{\max} t^*}{d}, \; V = \frac{V^*}{U^*_{\max}} = (\frac{V^*_{x^*}}{U^*_{\max}}, \frac{v_o}{U^*_{\max}}),$$
$$P = \frac{P^*}{\rho U^{*2}_{\max}}, \; \tau = \frac{\tau^*}{\rho U^{*2}_{\max}} \quad (6)$$

The basic velocity profile in non-dimensional form can be written as

$$U(y) = R_c \frac{y + \sinh^{-1}(R_c) - e^{-R_c y} - \coth(R_c)}{1 - \log(R_c^{-1} \sinh(R_c)) - R_c \coth(R_c)} \quad (7)$$

This basic solution depend only on the cross-flow Reynolds number, $R_c \, (= \frac{v_o d}{\nu})$, and it is identical to that established by Fransson and Alfredsson [3] for a Newtonian fluid. In addition, for R_c tend to zero, the basic velocity reduces to that of plane Poiseuille flow ($1-y^2$).

3. Linear stability analysis

To study the linear stability of this problem, we assume that the infinitesimal perturbations (**v** and p) are superimposed to the basic flow variables (**V** and P) as follows

$$V_x = U + u' ; V_y = v_o + v' ; \; P = P_b + p' \quad (8)$$

Using the stream function, ϕ, then the solutions can be sought into Fourier's modes as follows

$$(\phi, \, p) = [\varphi, p] e^{i\alpha(x-ct)} \quad (9)$$

where φ and p are, respectively, the complex amplitudes of the perturbations v' and p', and c $(=c_r+ic_i)$ is the complex wave speed and $i^2 = -1$. Also, α designates the wave number which is correlated with the wavelength, δ, by the relation $\alpha=2\pi/\delta$. The differential equation determining the stability is expressed by

$$\mathrm{Re}\, L_{os1} - \Delta\Delta\varphi - R_c \frac{\partial \Delta\varphi}{\partial y} + De \frac{\partial}{\partial t}\left[\mathrm{Re}\, L_{os1} + R_c \frac{\partial \Delta\varphi}{\partial y}\right] \quad (10)$$

where $R_e \, (= \frac{U_{\max} d}{\nu})$ represents the Reynolds number and L_{os1} is written as

$$L_{os1} = \frac{\partial \Delta\varphi}{\partial t} + U \frac{\partial \Delta\varphi}{\partial x} - \frac{\partial^2 U}{\partial y^2} \frac{\partial \varphi}{\partial x}$$

Equation (10) represents the classical Orr-Sommerfeld equation in which two additional terms are added. The first term is due to the cross-flow (R_c). The second one, is related to the non-dimensional relaxation time, λ, designated by Deborah number, $De \; (= \frac{U_{\max} d}{\lambda})$. The corresponding boundary conditions are:

$$\varphi(y = \pm 1) = \varphi,_y (y = \pm 1) = 0 \quad (11)$$

Eq. (10) associated to these boundary conditions (11) are resolved numerically using the Chebyshev spectral collocation method based on the most commonly used, N collocation points of Gauss- Labatto [9]. Under these conditions, our stability problem is reduced to an algebraic system with non-linear eigenvalues, c

$$E\varphi + cF\varphi + c^2 G\varphi = 0 \quad (12)$$

where E, F and G are three matrices containing De, Re, R_C, α and N. It should be noted that, this flow is linearly unstable if $c_i > 0$ [10]. The accuracy of the numerical code has been checked through comparison with the results of Orszag [11] for plane Poiseuille flow of a Newtonian fluid and the results obtained by Fransson and Alfredsson [3] for plane Poiseuille flow with cross- flow. All the numerical results presented in the present work are computed with 120 collocation points.

4. Discussions and conclusions

Our main purpose concerns the stability of the considered system on the basis of the typical behavior of the fastest growing rate of the most unstable mode, $\omega_i = \alpha c_i$, versus the wave number α under the combined effect of the cross-flow Reynolds and the Deborah numbers. In this case, we have assigned to the Reynolds number the value corresponding to the classical solution of Poiseuille flow ($Re_{cp}=5772.22$) [11]. The corresponding results are plotted in Figs. 2 and 3.

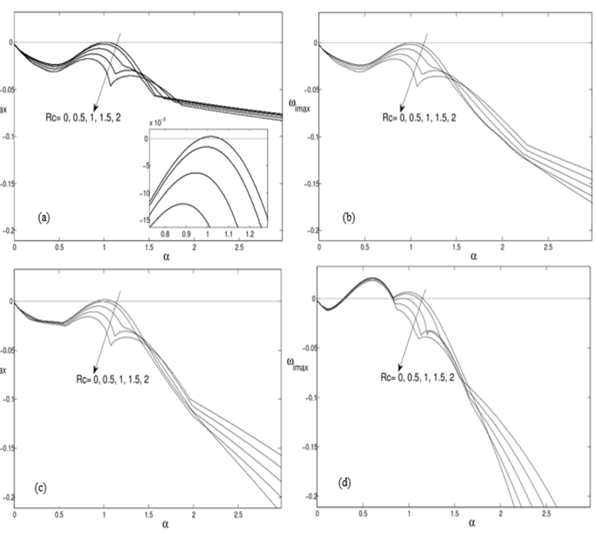

Fig .2: The fastest growing rate of ω vs wave number, α, for different values of Rc, (a): De=0; (b): De =0.01; (c): De =0.1; (d): De =0.5

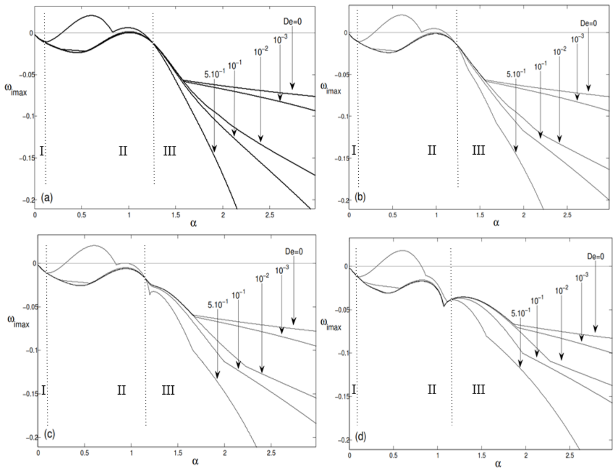

Fig .3: The fastest growing rate of ω vs wave number, α, for different values of De, (a): Rc=0; (b): Rc=0.5; (c): Rc=1; (d): Rc=2

Figs. 2(a-d) show the variations of ω_i with α for increasing values of R_C and different values of *De*. It can be seen that, for the Newtonian case and without cross-flow (*De=0* and $R_C=0$ [see Fig. 2-a]), the flow is always stable since ω_i is negative, An instability, however, is observed in a narrow region of the wave number, $0.9705<\alpha<1.0596$, where $\omega_i > 0$. This instability picture is changed in the presence of cross-flow where $R_C \neq 0$. Indeed, all curves in Figs. 2(a) show that, except an intermediate wave number region, an increase in R_C causes a weak decrease in the growth rate ω_i and a suppression of the instability zone occurred when $R_C = 0$. This result highlights a stabilizing effect of the cross flow at small and large wave number regions for a Newtonian fluid. This stabilizing effect is more pronounced near $\alpha_{cp} = 1.02056$, which correspond to the classical solution where $R_C = 0$ [11].
A different behavior is observed at large values of the wave number for the Maxwellian fluid, $De \neq 0$, [Figs. 3(b–d)] where the growth rate ω_i continuously increases with R_C. Similarly, to the Newtonian case, the cross-flow has the same effect on the stability of the system near α_{cp}. In addition, no effect of the number R_C is observed for the small values of the wave number especially for $De = 0.5$.

In order to discern the effect of the elasticity on the stability of the system, Figs. 3 (a-d) illustrate the evolution of the growth rate as a function of the wave number for increasing *De* and for several values of R_C. As one can notice, for small values of wave number (region I) there is no effect of the Deborah number while a strong stabilizing effect is observed for the high values of the wave number (region III). In addition, for $\alpha < \alpha_c$ (region II), where α_c is a wave number when an intersection of all the curves is occurred, the system is prone to stabilization (increase of ω_i) by increasing the fluid's elasticity up to *De=0.1*. Note that the cross flow number tends to decrease the value of α_c .as shown in Fig.4.

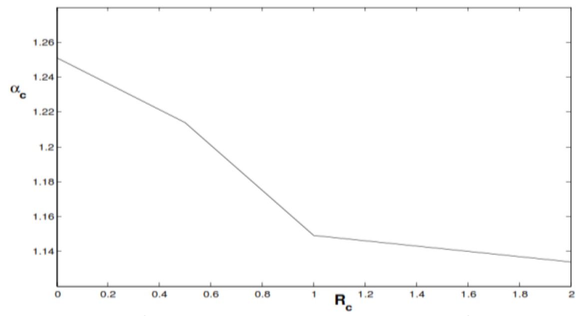

Fig .4: Variation of the critical wave number with cross-flow Reynolds number

To conclude, on the basis of the results presented in this paper, the cross-flow tends generally to stabilize the basic flow corresponding to a Newtonian fluid. This stabilizing effect is not conserved in the case of a Maxwell fluid, especially for short wavelengths where R_C has a destabilizing effect. On the other hand, the fluid's elasticity leads to enhance the stability of the flow in short wavelength and to delay this stability in the long wavelength region.

References

[1] A.S. Berman, *Concerning laminar flow in channels with porous walls*, J. Appl. Phys. 27 (1956) 1557.
[2] F. D. Hains, *Stability of plan Couette-Poiseuille flow with uniform cross flow*. Phys. Fluids 14, pp. 1620-1623, 1971
[3] D. Max Sheppard. *Hydrodynamic stability of the flow between parallel porous walls*. Phys. Fluids, 1972.
[4] J. H. M. Fransson and P. Henrik Alfredsson. *On the hydrodynamics stability of channel flow with cross flow*. Physics of Fluids, 15(2), pp. 436–441.
[5] M. Lamine and A. Hifdi, *stability analysis of plane couette-poiseuille flow with porous walls under transverse magnetic field*, Journal of Thermal Engineering Vol. 2, No. 4, pp. 807-813, July, 2016.
[6] Annable, T; Buscall, R; Ettelaie, R; Shepherd, P; Whittlestone, D., Langmuir, *Influence of surfactants on the rheology of associating polymers in solution*. 10(4), 1060 – 1070 (1994).
[7] Hoffmann, H., *Structure and Flow in Surfactant Solutions*, 578, 2 - 31 (1994). 2003.
[8] M. Riahi, S. Aniss, M. O. Touhami, S. S.Lami, *Centrifugal instability of pulsed Taylor-Couette flow in a Maxwell fluid*, Eur. Phys. J. E (2016) 39:82
[9] A. C. Weideman and S. C. Reddy, ACM Transactions on Mathematical Software. Vol. 26, 465, 2000
[10] A. Hifdi, M. Ouazzani-Touhami, and J. Khalid-Naciri, *Channel entrance flow and its linear stability*, J. Stat. Mech.: Theor. Exp., vol. P06003 (stacks.iop.org/JSTAT /2004/P06003), 2004.
[11] S. A. Orszag, *Accurate solution of the Orr-Sommerfeld stability equation*, J .Fluid Mech., vol. 50, no. 4, pp. 689-703, 1971.