Method of Fundamental Solutions for nonlinear elastic problems

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Abstract:

The aim of this work is to propose an approach which combines the meshless method called Method of Fundamental Solutions (MFS) and the Asymptotic Numerical Method (ANM) to solve two-dimensional nonlinear elastic problems. The ANM transforms the nonlinear problems into a sequence of linear problem which can be solved by MFS. This last consists to approach the solution of the linear problem by a linear combination of fundamental solutions with respect to some source points which are located outside the domain. Numerical results are presented for a simple traction square plate in large deformation framework to show the efficiency of the proposed algorithm.

Keywords: *Method of Fundamental Solutions, Asymptotic Numerical Method, nonlinear computation, Lagrangian formulation.*

1 Introduction

Many numerical methods have been proposed for the resolution of partial differential equations based on spatial discretization, which make it possible to obtain a finite number of unknowns. The Finite Element Method (FEM) is the standard technique for this kind of computations. But under the last fifteen years a new mesh free method has been under extensive research. In the family of meshless methods, we find the Method of Fundamental Solutions (MFS). The MFS is a collocation-type method, easy-to-use and has been widely applied to various engineering and science problems, in which the solution of the homogeneous problems is expressed by a linear combination of the fundamental solutions of the opertors with respect to some source points located outside the domain. This technque has been extended to non-homogeneous partial differential equations by using the Radial Basis Functions (RBF) for the determination of the particular solution. Then, the original problem is reduced to determining the unknown coefficients of the linear combination. The effectiveness of the MFS has been proven through its application on linear problems such as the Biharmonic, the Helmholtz, the Poisson, the linear elasticity Cauchy problem and the non-homogeneous linear elasticity equations. In 2009, Li et al. [3] have coupled the Method of Fundamental Solutions (MFS) with the Radial Basis Functions (RBF) and the Analog Equation Method (AEM), to solve nonlinear elliptic problems, the resolution of the obtained nonlinear equations is done by iterative algorithms. Tri et al. [4] have proposed in 2011 a coupling of the Asymptotic Numerical Method (ANM) and the Method of Fundamental Solutions associated with the RBF and AEM to solve the nonlinear Poisson problem in a limited framework, which permits one to compute a part of nonlinear response curves up to the radius of convergence. In 2012, Tri et al. [5] have presented a continuation algorithm able to compute the entire branch solution using the same basis functions. In this work, we use an algorithm which couples the Asymptotic Numerical Method with the Method of Fundamental Solutions for solving a nonlinear elasticity problem in two-dimensional framework in the context of large deformations. The Asymptotic Numerical Method (ANM) [2] is a family of algorithms that solves nonlinear problems thanks to an asymptotic development. Indeed, the ANM allows to search the branch solutions in the form of power series with respect to a path parameter "a". In this way, we transform the initial nonlinear problem into a recurring sequence of linear problems with the same tangent operator. The layout of this paper is as follows. In section 1, we present the equations governing the equilibrium of solids in nonlinear elasticity. Then we discuss the Asymptotic Numerical Method (ANM) in section 2. Afterward, we show how to solve the resulting linear problems using the coupling (MFS-RBF-AEM) in section 3. Finally, we validate our approach by a numerical application in nonlinear deformation computation of a square plate.

2 Mathematical formulation

The strong form of the equilibrium condition of a structure occupying the domain Ω with a boundary $\partial\Omega$, for an elasticity problem including geometric nonlinearity, is based on a Lagrangian formulation. Since the equations are formulated with respect to a reference configuration, the stationary problem of nonlinear elasticity is expressed by the following equations:

$$\begin{cases} \gamma & = & \frac{1}{2}\left({}^T\nabla(U) + \nabla(U) +^T \nabla(U)\nabla(U)\right) \\ S & = & \mathbb{C} : \gamma \\ \Pi & = & \mathbb{F} \,.\, S \\ \nabla \,.\, \Pi + b & = & 0 \end{cases} \tag{1}$$

where Π is the first Piola-Kirchhoff stress tensor associated with a point of the domain Ω in its reference configuration, b is a body force term, $\mathbb{F}$ is the deformation gradient tensor defined by $\mathbb{F} = I + \nabla U$, where I is the identity tensor, S, γ, U and $\mathbb{C}$ are respectively the second Piola-Kirchhoff stress tensor, the Green-Lagrange strain tensor, the displacement field and the fourth-order stiffness tensor. The equations defined in (1) are complemented by the boundary conditions on the boundary $\partial\Omega$. This boundary $\partial\Omega$ is subjected to the displacements and the traction data on the disjoint complementary parts of the border $\partial\Omega_u$ (Dirichlet Boundary conditions) and $\partial\Omega_f$ (Neumann Boundary conditions), respectively ($\partial\Omega$=$\partial\Omega_u \cup \partial\Omega_f$ and $\partial\Omega_u \cap \partial\Omega_f = 0$). The boundary conditions are given by:

$$\begin{cases} \Pi \cdot n & = & T^d & in\ \partial\Omega_f \\ U & = & U^d & in\ \partial\Omega_u \end{cases} \tag{2}$$

where n is the unit normal at $\partial\Omega$, U^d and T^d are respectively displacements and the traction data.

Our study will be limited to the two-dimensional structures framework $U \equiv \{U\} = \langle u_1 \quad u_2 \rangle$.

3 ANM solution strategy

In this section, we apply the ANM algorithm to the nonlinear problem (1). The development in power series makes it possible to transform the nonlinear problem into a sequence of linear problems. Recall that within the framework of ANM, it is preferable to formulate the problem to be solved in a quadratic form, which is practical to obtain the reccurence formula in a easy way when generating linear problems. The fundamental physical quantities of the problem can be represented by the mixed vector $\mathbb{U} = \{\Pi, S, \gamma, U\}$ and a loading parameter λ which allows to determine all the equilibrium positions of the problem. In this technique, the variables $\mathbb{U}$ and λ are developed using a asymptotic expansion truncated at order P with respect to a path parameter "a" in the neighbourhood of a known starting solution ($\mathbb{U}_0,\ \lambda_0$), thus, we can write:

$$\begin{Bmatrix} \mathbb{U} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbb{U}_0 \\ \lambda_0 \end{Bmatrix} + \sum_{i=1}^{P} a^i \begin{Bmatrix} \mathbb{U}_i \\ \lambda_i \end{Bmatrix} \tag{3}$$

By injecting the developments (3) into the problem (1) and by regrouping the terms of the same power of "a", we obtain a recurring sequence of linear problems having the same tangent operator for all orders depending on the geometry and the initial stress S_0. This procedure involves calculating the i^{th} order of the series (3) with the i^{th-1} orders previously calculated. An additional equation called the closing equation must be added to define the path parameter "a". Several choices are possible. The most used is a pseudoarc-length parametrization. It corresponds to the projection of the pair $(U - U_0, \lambda - \lambda_0)$, ie, the displacement and the load parameter, on the tangent direction U_1 , λ_1.

$$a = \langle U - U_0 \rangle \{U_1\} + (\lambda - \lambda_0)\lambda_1 \tag{4}$$

The step of development in series is continued with a continuation step which makes it possible to obtain the whole solution branch [2]. The continuation technique is related to the determination of the convergence radius "a" of the series (3). In this article, the validity range of the solution is defined by the maximal value a_{max} of the control parameter "a". Requiring that the relative difference between the displacements at two consecutive orders must be smaller than a given parameter ε leads to

$$a_{max} = \left(\varepsilon \frac{\|U_1\|}{\|U_P\|}\right)^{\dfrac{1}{P-1}} \tag{5}$$

4 MFS-MPS spatial discretization

A meshless method was presented, which couples the Method of Fundamental Solutions (MFS) with Radial Basis Functions (RBF) and the Analog Equation Method (AEM), to solve the previous linear problems with the tangent operator computed at the starting point, in this method, the AEM is used to convert the governing equation (1) which expanded with power series into a corresponding linear inhomogeneous equation, so that a simpler fundamental solution can be employed. Then, the RBFs and the MFS are, respectively, used to construct the expressions of particular and homogeneous solution, from which the main unknown of the problem is approximated by a superposition of the homogeneous solution and of the particular solution.

$$\begin{aligned} \{U(M_i)\} = & \textstyle\sum_{j=1}^{N_s} \left[\widehat{U}^h(M_i, Q_j)\right] \begin{Bmatrix} \alpha_j^h \\ \beta_j^h \end{Bmatrix} \\ & + \textstyle\sum_{j=1}^{N} \left[\widehat{U}^p(M_i, M_j)\right] \begin{Bmatrix} \alpha_j^p \\ \beta_j^p \end{Bmatrix} \end{aligned} \tag{6}$$

Where $Q_j(X_1^j, X_2^j)$ and $M_i(x_1^i, x_2^i)$ are the coordinates of the N_s source points taken on a fictitious boundary Γ_f and coordinates of the N points of the domain respectively.
$[\widehat{U}^h(M_i, Q_j)]$ represents the fundamental solutions matrix of the linear elasticity operator in two-dimension. These fundamental solutions are given in [1].
$[\widehat{U}^p(M_i, M_j)]$ represents the particular solutions matrix. For a Radial Basis Function (RBF) of the type Thin Plate Spline (TPS) $r^n log(r)$ the particular solution is given in [1].
After satisfying all equations of the original problem (1) and (2) at collocation points, a system of equations is represented by the unknown coefficients can be obtained using SVD(Singular Value Decomposition) solver

5 Numerical application

To evaluate the performance and the robustness of our algorithm, we propose to apply it to the nonlinear problem of a square plate of side $L = 100mm$ in plane stress, the structure is subjected at both ends of a stress $\sigma_{11} = \lambda\sigma_0$ with $\sigma_0 = 100MPa$ and λ increases gradually to determine all equilibrium positions. This structure is elastic, homogeneous and isotropic of Young's modulus $E = 200.10^3 MPa$ and Poisson coefficient $\nu = 0.3$. In this example, the fictitious boundary is a circle of radius $R = 150mm$. We take $N = 121$ points distributed over the domain occupied by the plate and $(N_s = 40)$ is the number of points on the fictitious boundary chosen equal to the number of the boundary points in the domain (see Figure 1).

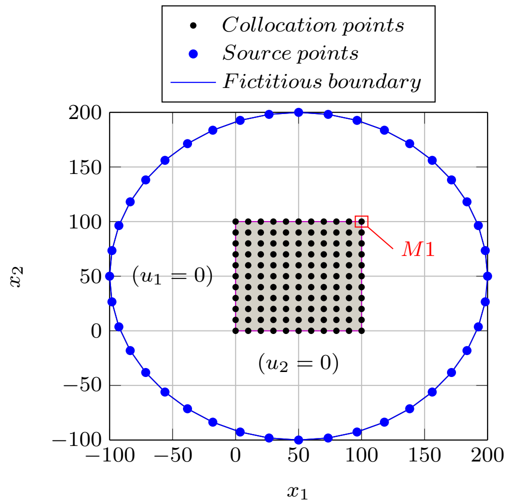

Figure 1: domain Ω, collocation points and source points

The results obtained from this example are shown in figure.2. In this figure, we represent the obtained solution (Load-Displacement) of the point M1 illustrated in figure.1. Compared to the finite elements method , we remark that the meshless method and the asymptotic method give a good solution for the nonlinear problem.

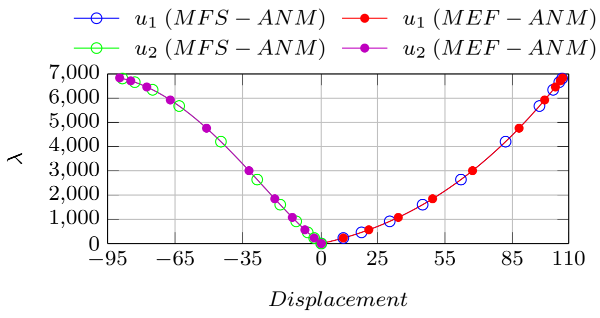

Figure 2: Load-displacement curve for 10 branches with a truncation order $P = 15$ and a tolerance of 10^{-6}

6 Conclusion

In this work, we have extended the Method of Fundamental Solutions to nonlinear elasticity problems by associating it with Asymptotic Numerical Methods. The numerical results for two-dimensional problems show the efficiency of the proposed algorithm using a comparison with the Finite Element Method coupled with the ANM. The proposed technique is tested successfully. Work is currently in progress to study the instabilities of the structures by introducing the bifurcation indicators and the Pade approximants.

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