THIRD DISCONTINUOUS GALERKIN METHOD FOR TRANSCRITICAL FLOW

A. AMAHMOUJ[1], E. M. CHAABELASRI [1,2], N. SALHI[1]

1. LME, Faculté des Sciences, BP 717, 60000 Oujda, Maroc + a.amahmouj@gmail.com
2. ENSA, BP 03, Ajdir Al-Hoceima, Maroc + chaabelasri@gmail.com

Abstract :

This paper presents a new one-dimensional (1D) third-order Runge–Kutta discontinuous Galerkin (RKDG3) scheme for shallow flow. The shallow water equations that adopt water level as a flow variable are solved by an RKDG3 scheme to give piecewise linear approximate solutions, which are locally defined by an average coefficient and a slope coefficient. Extra numerical enhancements are proposed to amend the local coefficients associated with water level in order to maintain the well-balanced property of the RKDG3 scheme for real applications. Friction source terms are included and evaluated using splitting implicit discretization, implemented with a physical stopping condition to ensure stability. steady and unsteady benchmark tests with/without friction effects are considered to demonstrate the performance of the present model.

Key words. Shallow water flows, Runge-Kutta Discontinuous Galerkin.

1. Introduction

A number of engineering and environmental applications are characterized by free surface flows in which the typical horizontal length scale is large compared with the vertical scale. In this case, the flow can be approximated by the shallow water equations (SWE).

Numerical simulation of free surface flows usually involves the solution of some difficult tasks, such as the capture of sharp discontinuities due to hydraulic jumps or bores in shallow flows, the flood-wave propagation over an initially dry land and its alternate recession, with consequent drying of the bottom, or the development of fast-moving flows over an irregular topography. In last few decades, computational models based on the solution of the SWE have been developed to study a variety of free-surface flows with finite difference, finite volume and finite element methods (see, e.g. [2] and references therein). More recently, discontinuous Galerkin methods (DG) have appeared as an alternative to finite volume or high-order finite difference schemes, with a number of advantages: these allow efficient p- and h-adaptivity, local grid refining and parallel computations.

In this work, we propose the numerical solution of the SWE with the DG method and expanded it with numerical techniques in order to satisfy good shock capturing due to dam break. The accuracy of the method is evaluated by comparison against a survey of analytical test.

2. Governing shallow water model

The SWE for long-wave propagation may be derived by intentions by assuming negligible vertical particle acceleration and thus hydrostatic distribution. Including bed slope and friction effects the SWE may be adequate for describing a wide range of 1D and 2D shallow flow problems. In recent years, it has been generally accepted that the use of the surface water elevation η(x, t) instead of the water depth h(x, t) as a flow variable in the mathematical shallow water model may lead to a well-balanced numerical scheme that preserves the solution of lake at rest at the computational level [8, 9, 13, 14]. Furthermore, adopting η as a flow variable improves the quality of a slope limiting process for RKDG method [7, 12]. In a matrix form, the 1D conservation laws of the nonlinear hyperbolic SWE may be written as [8, 14]

$$U_t + F_x = S \qquad (1)$$

In which, $U=[\eta,q]^T$. $q(x,t)$ is the unit-width discharge. $\eta(x,t)=h(x,t)+z(x)$ with $z(x)$ being the ground elevation. $u(x,t)=q(x,t)/h(x,t)$ is the depth-average velocity. t and x denotes, respectively, the time and space coordinate. $F=[q,q^2/h+g(\eta^2-2\eta z)/2]^T$ is the flux vector such that $J=\partial F/\partial U$ has two real eigenvalues $\lambda^{1,2}=u\pm c$ and two associated real eigenvectors $e^{1,2}=[1,\lambda^{1,2}]^T$, where $c=\sqrt{gh}$ is the shallow wave speed and g is the gravitational acceleration. Obviously, the system is strictly hyperbolic if $h\neq 0$ [11]. $S=S_b+S_f$ is the vector containing the bottom topography and friction source terms. $S_b=[0,S_b]^T$ with $S_b=g\eta S_0$ and $S_0=-\partial Z/\partial x$. $S_f=[0,S_f]^T$, with $S_f=-C_f u|u|$ with $C_f=gn_M^2/h^{1/3}$ and n_M being the Manning coefficient.

3. Discontinuous Galerkin method formulation

The 1D domain, on which the governing equations are solved, has a length of L and is divided by the interface points $0 = x_{1/2} < x_{3/2} < ... < x_{N+1/2} = L$ into N uniform intervals(cells). The size of an arbitrary cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ is $\Delta x = x_{i-1/2} - x_{i+1/2}$ and the nodal points is at $x_i = (x_{i+1/2} + x_{i-1/2})/2$. When solving the conservation laws of the SWE(1) using a finite element Galerkin method, a kth order approximation of the flow variables $U_h(x,t) = [\eta_h(x,t), q_h(x,t)]^T$ is sought, which belongs to the finite dimensional space $V_h = P : P|_{I_i} \in P^k(I_i)$ and $P^k(I_i)$ is the polynomial space in I_i of degree at most k [6]. The approximation gaves $(k+1)th$ order of accuracy in space. In order to derive the RKDG discretized governing equation, a test function $v_h \in V_h$ is introduced to (1), which is then integrated over I_i. Subsequently, integrating by part the flux derivative term gives a weak form to (1)

$$\int_{I_i} \partial_t U_h(x,t)\upsilon_h(x)dx + [F(U_h(x_{i+1/2},t)\upsilon_h(x_{i+1/2}) - F(U_h(x_{i-1/2},t)) \upsilon_h(x_{i-1/2}) - \int_{I_i} F(U_h(x,t))\upsilon_h(x)dx] = \int_{I_i} S(U_h(x,t))\upsilon_h(x)dx \quad (2)$$

A finite number of local basis functions are introduced using the $L^2 - orthogonal$ basis of Legendre polynomials [5] on I_i to locally expand the flow variables. The expanded flow variables are then substituted into the weak formulation (2) and a test function is chosen to specifically coincide with a basis function. As a result the DG space discretisation of (1) reduces to an independent a second- order scheme is considered for practical simulations and only two degrees of freedom (k.e. $k = 1$) are required [7]. Therefore, at each cell, the local linear approximate solution $U_h(x,t)$ is determined by two polynomial coefficients $\{U_i^0(t), U_i^1(t)\}$, i.e.

$$U_h(x,t)|_{I_i} = U_i^0(t) + 2U_i^1(t)(x - x_i)/\Delta x \quad (\forall x \in x_{i+1/2}) \quad (3)$$

where $U_i^0(t)$ is the vector of the averaged flow variables at the cell centres and $U_i^1(t)$ contains the corresponding slope coefficients. The degrees of freedom $\{U_i^0(t), U_i^1(t)\}$ are stored and evolved locally in each cell, starting by the Legendre basis [5]. The initial values of averaged for flow variables is written as (refer [7] for further details)

$$U_i^{0,1}(0) = \frac{U_0(x_{i+1/2}) \pm U_0(x_{i-1/2})}{2} \quad (4)$$

The detached ODEs $(dU_i^{0,1}(0)/dt)$ for the degrees of freedom are then spatially approximated by the discrete operators $L_i^{0,1}$

$$dU_i^{0,1}/dt = L_i^{0,1}(U_{i-1}^{0,1}, U_i^{0,1}, U_{i+1}^{0,1}) \quad (5)$$

The local operator may be referred to as DG2 operators and, by choosing the correct Gaussian local points, written as

$$L_i^0 = -\frac{1}{\Delta x}[\overline{F}_{i+1/2} - \overline{F}_{i-1/2} - \Delta x S_b(U_i^0)] \quad (6)$$

$$L_i^0 = -\frac{3}{\Delta x}\left\{\overline{F}_{i+1/2} + \overline{F}_{i-1/2} - F_b(U_i^0 + \frac{U_i^1}{\sqrt{3}}) - F_b(U_i^0 - \frac{U_i^1}{\sqrt{3}}) - \frac{\Delta x\sqrt{3}}{6}\left[S_b(U_i^0 + \frac{U_i^1}{\sqrt{3}}) - S_b(U_i^0 - \frac{U_i^1}{\sqrt{3}})\right]\right\} \quad (7)$$

The local approximate solution can fall discontinuous at the interface points $x_{i+1/2}$ and $x_{i-1/2}$. Therefore, the fluxes therein $F(U_h(x_{i\pm1/2},t))$ are replaced by the Godunov-type numerical fluxes $\overline{F}_{i\pm1/2} = F(U_{i\pm1/2}^-, U_{i\pm1/2}^+)$ as in a finite volume scheme. $U_{i\pm1/2}^{\pm}$, respectively, denote the face values of the flow variables at the left and right-hand sides of cell interface $x_{i\pm1/2}$, which are also referred to as Riemann states. Therefore $\overline{F}$ represents the two argument numerical flux functions defined by the Riemann states and can be evaluated by a Riemann solver. It should be noted that (6) and (7) only contain the topographic source terms and treatment of the friction source terms.

When reconstructing the Riemann states and hence evaluating the interface fluxes, it is essential to restrict the local maximum slope of the flow variables by means of a slope limiter. The slope limiting step damps the spurious oscillations would otherwise order numerical scheme. The minmod slope limiter [5, 15] is used in this work for better stability

$$\overline{\overline{U}}_i^1 + F_x = \min\bmod(U_i^1, U_{i+1}^0 - U_i^0, U_i^0 - U_{i-1}^0) \quad (8)$$

$\overline{\overline{U}}_i^1$ denote the limited vector of slope coefficients and accordingly the face values of the flow variables are evaluated by $\overline{\overline{U}}_{i\pm1/2}^{\pm} = U_i^0 \pm \overline{\overline{U}}_i^1$. When flow calculation occurs in a dry cell or a wet cell adjacent to a dry cell, minmod limiting process in (8) is deactivated to give $\overline{\overline{U}}_i^1 = U_i^1$ in order to maintain better numerical stability.

A two-stage monotonicity-preserving RK time integration [10] is implemented to achieve second-order accuracy in time

$$(U_i^{0,1})^{n+1/2} = (U_i^{0,1})^n + \Delta t(L_i^{0,1})^n \quad (9)$$

$$(U_i^{0,1})^{n+1} = \frac{1}{2}\left[(U_i^{0,1})^n + (U_i^{0,1})^{n+1/2} + \Delta t(L_i^{0,1})^{n+1/2}\right] \quad (10)$$

The stability of the above explicit RKDG2 scheme is controlled by the CFL criterion and the CFL number should be, at most, equal to 0.333 to ensure stability [5]. The CFL number is set to 0.3 for all the test cases in this work. as previously, this tight restriction on time step represents the main drawback of an explicit RKDG method.

4. Results and discussions

In this test, a wide rectangular channel with a horizontal and frictionless bed was considered. The resulting governing equations reduced to conservation laws without source term. The channel was 1m in length. The dam was located at 0.5m from upstream. Initially, the upstream water depth was 1m and downstream water depth was set to 0.5m. Results are shown in Figure 1. The first one represent the free surface water depth and the second is the water velocity at 0.1s. The numerical solutions of water depth and velocity are in good agreement with exact solution and finite volume Roe scheme. The results show that all tow numerical flux functions capable of capturing the shock within the domain. The zooming snapshots performed show that the RKDG flux function has the best solution among the evaluated Roe flux function.

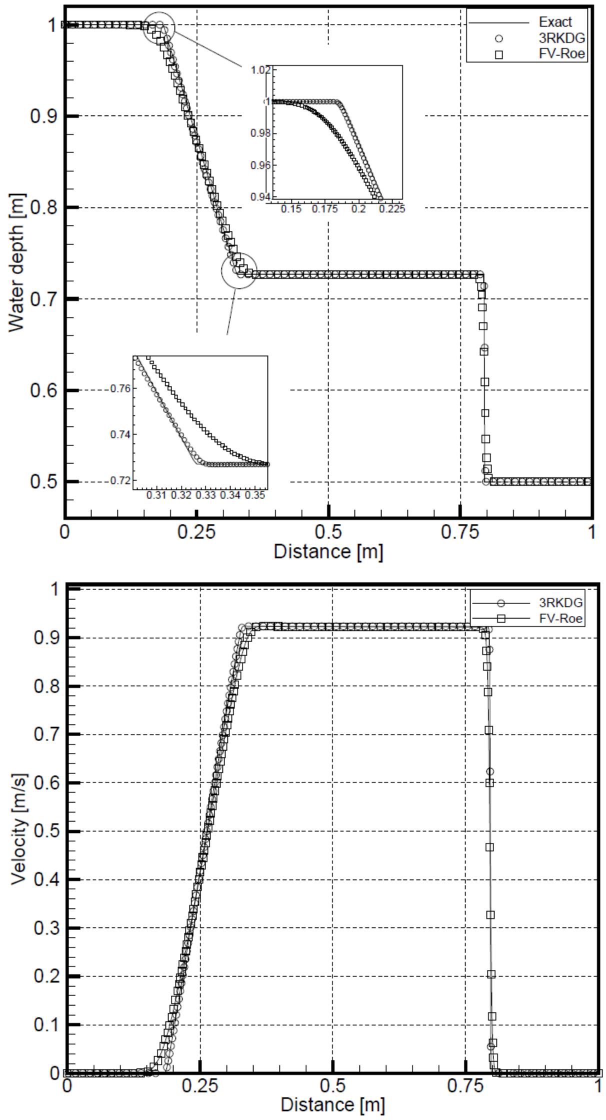

Figure 1: Dam break problem: Free water surface profile (Top) and flow velocity (bottom).

References

[1] R. GLOWINSKI, J. L. LIONS, R. TREMOLIERES, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, (1981) .

[2] B. Cockburn, CW. Shu, *Runge–Kutta Discontinuous Galerkin Methods for Convection-Dominated Problems,* Journal of Scientific Computing (2001).

[3] B. Cockburn CW. Shu, *The Runge–Kutta local projection P1-discontinuous Galerkin method for scalar conservation laws*, RAIRO Modél. Math. Anal. Numér. 25, 337–361 (1991).

[4] G Kesserwani, Q Liang, J Vazquez, R Mosé, *Well- balancing issues related to the RKDG2 scheme for the shallow water equations*, International journal for numerical methods in fluids 62 (4), 428-448, (2010).

[5] Q. Liang, A.G.L. Borthwick, *Adaptive quadtree simulation of shallow flows with wet–dry fronts over complex topography*, Comput. Fluids 38(2), 221–234, (2009).

[6] S. Noelle, N. Pankratz, G. Puppo, and J. R. Natvig, *Well-balanced finite volume schemes of arbitrary order of accuracy for shallow water flows*, Journal of Computational Physics 213 , 474–499, ((2006).

[7] C. W. Shu and S. Osher, *Efficient implementation of essentially non-oscillatory shock- capturing schemes*, J. Comput. Phys. 77. 439 (1988).

[8] Y. Xing and CW. Shu, *A New Approach of High OrderWell-Balanced Finite Volume WENO Schemes and Discontinuous Galerkin Methods for a Class of Hyperbolic Systems with Source Terms*, COMMUNICATIONS IN COMPUTATIONAL PHYSICS; 1(1):101–35 (2006).

[9] J.G. Zhou, D.M. Causon, C.G. Mingham, D.M. Ingram, *The Surface Gradient Method for the Treatment of Source Terms in the Shallow-Water Equations*, Journal of Computational Physics;168(1):1–25, (2001).

[10] Q. Liang, F. Marche, *Numerical resolution of well-balanced shallow water equations with complex source terms*, Advances in water resources, 32:873–84, (2009).

[11] P. K. Sweby, *High resolution schemes using flux limiters for hyperbolic conservation laws*, SIAM journal on numerical analysis, 21(5):995–1011 (1984).