

NON-HOLONOMIC MECHANICS: A GEOMETRICAL TREATMENT OF AUTONOMOUS BICYCLE MOTION

Z. CHEN¹, M. RHAZI², M. AGHFIR², A. ES-SAIDI³, S. E. FAIK², S. HADDOUT³

¹University of British Columbia, 1935 Lower Mall, Vancouver, BC Canada V6T1X1, Canada.

²Department of Physics, Ecole Normale Supérieure, B.P 2400, 40000 Marrakech, Morocco.

³Department of Physics, Faculty of Science, Ibn Tofail University, B.P 242, 14000 Kenitra, Morocco.

Abstract

A geometrical theory of general nonholonomic mechanical systems on fibred manifolds and their jet prolongations, based on so-called Chetaev-type constraint forces, was developed in 1990s by Krupková. The relevance of this theory for general types of nonholonomic constraints, not only linear or affine ones, was then verified on appropriate models. Frequently considered constraints on real physical systems are based on rolling without sliding, i.e. they are holonomic, or semi-holonomic, i.e. integrable. Moreover, there exist some practical examples of systems subjected to true (non-integrable) nonholonomic constraint conditions. On the other hand, the equations of motion of a bicycle are highly nonlinear and rolling of wheels without slipping can only be expressed by nonholonomic constraint equations. In this paper, the geometrical theory is applied to the abovementioned mechanical problem using the above mentioned Krupková approach. Both types of equations of motion resulting from the theory-deformed equations with the so-called Chetaev-type constraint forces containing Lagrange multipliers, and reduced equations free from multipliers are found and discussed.

Key-words: *Nonholonomic mechanics; autonomous bicycle; Krupková approach; Lagrange multipliers; reduced equations of motion.*

1. Introduction

The motion of mechanical systems is frequently subjected to various constraint conditions, holonomic or nonholonomic. Nonholonomic constraints lead typically to nonlinear equations of motion of the constrained system. While theories of holonomic or some special types of linear non-holonomic constraints are already well elaborated for quite general situations, various theoretical approaches to general non-holonomic mechanics occur up to now, from the physical point of view on the one hand, and from the geometrical point of view on the other. The geometrical theory used in the presented study was presented for first order mechanical problems in [1] and then generalized for higher order case in [2] brings an appropriate tool for constructing certain type of equations of motion of nonholonomic

mechanical systems subjected to quite general constraints. The main physical idea of the theory is based on the concept of Chetaev-type constraint forces introduced in analogy to “classical” Chetaev forces. Using equations of constraints a special canonical distribution on the first jet prolongation of the underlying manifold can be constructed. Then first prolongations of admissible trajectories of the constrained motion are just integral sections of this distribution. By adding Chetaev-type forces to equations of motion, a dynamical form of the constrained problem is obtained and deformed equations of motion are constructed. These equations together with constraint conditions give the system of differential equations for unknown constrained trajectories and Lagrange multipliers. Another possible approach to the problem within the same theory starts from its description by the so-called Lepage class of forms instead the dynamical form itself. The Lepage class is, of course, closely related to the dynamical form, and it is obtained by the factorization of modules of forms by special submodules irrelevant from the point of view of the problem. Even though the corresponding constraint is semi-holonomic and thus it could be in principle treated by classical methods of Lagrange multipliers (for details concerning the method in general see e.g. the classical textbook of analytical mechanics [2]), the direct application of Krupková’s geometrical theory is very effective in this situation. On the other hand, a great interest has been devoted towards bicycle modeling as it is a mechanical system characterized by nonholonomic constraints. On the other hand, the bicycle is probably the most common mode of transportation in the world, next only to walking, and, starting from some pioneering papers at the end of the nineteenth century, many researchers have tried to find proper equations to describe the dynamic of this system. Mainly, it is possible to distinguish between two different approaches: the first obtains the motion equations using the Newton’s laws, while the second studies the system from a Lagrangian or Hamiltonian point of view. So far, the greatest part of the existing literature has been dedicated to models with lots of simplifications, even if these have been capable to explain the dynamical characteristics of the bicycle. For example, linearized equations of motion are commonly introduced in order to cope more easily with the problem. The aim of this paper is to use the geometrical theory for obtaining non-linear equations of

motion of the above exposed mechanical problem, using the above mentioned Krupková approach for a complex mechanical system (high number of degree of freedom) and find their solution in some particular cases. This is made in the last section, where both respective sets of equations of motion (reduced and deformed) are derived.



Figure 1. A photo of a real bicycle.

2. Geometrical theory of mechanical systems

For more details and proofs of geometrical concepts of the theory see [1]. The detailed theoretical background can be found in [2].

A non-holonomic constrained mechanical system is defined on the $(2m+1-k)$ -dimensional constrained sub-manifold $\wp \subset J^1Y$ fibered over Y and given by k equations $(1 \leq k \leq m-1)$: $f^i(t, q^\sigma, \dot{q}^\sigma) = 0$ such that $\text{rank}(\partial f^i / \partial \dot{q}^\sigma) = k$ $1 \leq i \leq k$. It is evident that only admissible trajectories for a non-holonomic mechanical system are such sections $\gamma : I \ni t \rightarrow Y$ for which $J^1\gamma(t) \in \wp$ for all $t \in I$, i.e $f^i \circ J^1\gamma = 0$ for $1 \leq i \leq k$ (the so-called \wp -admissible sections).

1-A section γ of (Y, π, X) is a path of the deformed system $[\alpha_\Phi]$ if and only if for every π_1 -vertical vector field η belonging to \mathfrak{S} holds

$$f^i \circ J^1\gamma = 0, (A'_l + B'_{ls} \ddot{q}^s) \circ J^2\gamma = 0 \quad (1)$$

Where:

$$A'_l = \left(\begin{array}{c} A_l + \sum_{j=1}^k A_{m-k+j} \frac{\partial g^j}{\partial \dot{q}^l} + \dots \\ \dots \sum_{i=1}^k \left(\begin{array}{c} B_{l,m-k+i} + \dots \\ \dots \sum_{j=1}^k B_{m-k+j,m-k+i} \frac{\partial g^j}{\partial \dot{q}^l} \end{array} \right) \times \dots \\ \dots \left(\begin{array}{c} \frac{\partial g^i}{\partial t} + \frac{\partial g^i}{\partial q^\sigma} \dot{q}^\sigma \end{array} \right) \end{array} \right) \circ t \quad (2)$$

$$B'_{ls} = \left(\begin{array}{c} \left[\begin{array}{c} B_{l,m-k+i} \frac{\partial g^i}{\partial \dot{q}^s} + \dots \\ \dots B_{m-k+i,s} \frac{\partial g^i}{\partial \dot{q}^l} \end{array} \right] \dots \\ \dots \left(\begin{array}{c} B_{m-k+j,m-k+i} \frac{\partial g^j}{\partial \dot{q}^l} \frac{\partial g^i}{\partial \dot{q}^s} \end{array} \right) \end{array} \right) \circ t \quad (3)$$

Relations (1) represent the system of reduced equations for m unknown functions $q^\sigma \gamma$ (k of them are first order and $(m-k)$ second order ordinary differential equations).

Physical approach is based on Chetaev-type constraint forces. Such a force is given by the constraint itself, in analogy with holonomic situations. It is expressed by the dynamical form:

$$\Phi = \Phi_\sigma \omega^\sigma \wedge dt = \mu_i \frac{\partial f^i}{\partial \dot{q}^\sigma} \wedge dt \quad 1 \leq i \leq k \quad (4)$$

where functions $\mu_i(t, q^\lambda, \dot{q}^\lambda)$ are Lagrange multipliers. Note that such dynamical form satisfies the generalized principle of virtual work $i_\eta \Phi|_U = 0$ for every π_1 -vertical vector field η belonging to the constraint distribution \mathfrak{S}_U , $\mathfrak{S}_U^0 = \text{span}\{\varphi^i, df^i, 1 \leq i \leq k\}$, $U, U \cap Q \neq \emptyset$ being an open set of a chart on J^1Y . Denote:

$$\alpha_\Phi = \left[A_\sigma - \mu_i \frac{\partial f^i}{\partial \dot{q}^\sigma} \right] \omega^\sigma \wedge dt + \dots \dots B_{\sigma\nu} \omega^\sigma \wedge \dot{q}^\nu + F_{\sigma\nu} \omega^\sigma \wedge \omega^\nu \quad (5)$$

The equivalence class $[\alpha_\Phi]$ is called the deformed mechanical system.

A \wp -admissible section γ of (Y, π, X) is called a path of $[\alpha_\Phi]$ if $(E_\sigma - \Phi_\sigma) \circ J^2\gamma = 0$. The following proposition holds :

2-A section γ of Y is a path of the deformed system $[\alpha_\Phi]$ if and only if for every π_1 -vertical vector field η on J^1Y it holds

$$A_\sigma + B_{\sigma\nu} \ddot{q}^\nu = \mu_i \frac{\partial f^i}{\partial \dot{q}^\sigma} \text{ and } f^i \circ J^1\gamma = 0 \quad (6)$$

System (6) is given by k first order and m second order ordinary differential equations for unknown functions μ_i and $q^\sigma \gamma$ and it represents the deformed equation. In the following section we apply the obtained equations (1 and 6) obtained for general non-holonomic mechanical system to the example of bicycle system.

3. Application: a Bicycle dynamic motion

There are seven degrees of freedom of the corresponding unconstrained mechanical system i.e. $m=7$. Thus, the fibered manifold of the problem is $(\square \times \square^7, pr_1, \square)$ where pr_1 is the cartesian projection on the first factor. We choose the fibered chart on Y as (V, ξ) where V is an open set $V \subset Y$ and $\xi_1 = (t, X, Y, \phi_f, \phi_r, \theta, \chi, \psi)$. The associated chart on the base is (pr_1, Φ) , $\Phi = (t)$ where t is the time coordinate, and associated fibered chart on the base is (pr_1, \dots) , where t is the time coordinate, and associated fibered chart on $J^1Y = \square \times \square^7 \times \square^7$ is (V_1, ξ_1) , $V_1 = pr^{-1}(V, \xi)$, $\xi_1 = (t, q^\sigma, \dot{q}^\sigma)$, $1 \leq \sigma \leq 7$ i.e. $\xi_1 = (t, X, Y, \phi_f, \phi_r, \theta, \chi, \psi, \dot{X}, \dot{Y}, \dot{\phi}_f, \dot{\phi}_r, \dot{\theta}, \dot{\chi}, \dot{\psi})$.

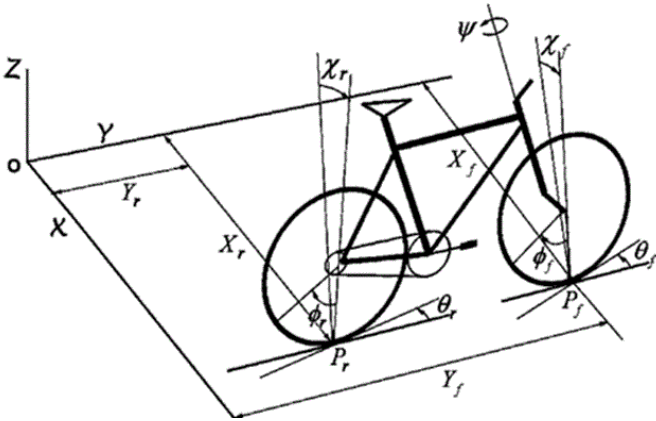


Figure 2. All parameters position for bicycle motion

Constrained mechanical system-reduced equations:

Computing the coefficients A_i according to equation (2) and coefficients B'_s according to equation (3). The reduced equations are of the form (Eqs: 8 and 9):

$$\left[\begin{array}{l} A'_1 + B'_{11} \ddot{\phi}_r + B'_{12} \ddot{\phi}_f + B'_{13} \ddot{\theta}_r + \dots \\ \dots B'_{14} \ddot{\theta}_f + B'_{15} \ddot{\psi} \\ A'_2 + B'_{22} \ddot{\phi}_f + B'_{21} \ddot{\phi}_r + B'_{23} \ddot{\theta}_r + \dots \\ \dots B'_{24} \ddot{\theta}_f + B'_{25} \ddot{\psi} \\ A'_3 + B'_{33} \ddot{\theta}_r + B'_{31} \ddot{\phi}_r + B'_{32} \ddot{\phi}_f + \dots \\ \dots B'_{34} \ddot{\theta}_f + B'_{35} \ddot{\psi} \\ A'_4 + B'_{44} \ddot{\theta}_f + B'_{41} \ddot{\phi}_r + B'_{42} \ddot{\phi}_f + \dots \\ \dots B'_{43} \ddot{\theta}_r + B'_{45} \ddot{\psi} \\ A'_5 + B'_{55} \ddot{\psi} + B'_{51} \ddot{\phi}_r + B'_{52} \ddot{\phi}_f + \dots \\ \dots B'_{53} \ddot{\theta}_r + B'_{54} \ddot{\theta}_f \end{array} \right] \circ J^2 \gamma' = 0 \quad (7)$$

$$\begin{aligned} & \ddot{\theta}_r - \frac{R}{l_r + l_f} \ddot{\phi}_r (\cos \theta_r \tan(\theta_r + \psi \cos \lambda) - \sin \theta_r) \dots \\ & \dots - \frac{h_r \sin \lambda - l_r \cos \lambda}{l_r + l_f} \ddot{\psi} = 0 \\ & \ddot{\theta}_f - \frac{R}{l_r + l_f} \ddot{\phi}_f (\cos \theta_f \tan(\theta_f + \psi \cos \lambda) - \sin \theta_f) \dots \\ & \dots - \frac{h_f \sin \lambda - l_f \cos \lambda}{l_r + l_f} \ddot{\psi} = 0 \\ & \ddot{\phi}_f - \frac{\cos \theta_f}{\cos(\theta_f + \psi \cos \lambda)} \ddot{\phi}_r = 0, \\ & \ddot{\phi}_r - \frac{\cos(\theta_r + \psi \cos \lambda)}{\cos \theta_r} \ddot{\phi}_f = 0 \\ & \ddot{X}_r + R \ddot{\phi}_r \sin \theta_r = 0, \quad \ddot{X}_f + R \ddot{\phi}_f \sin \theta_f = 0 \\ & \ddot{Y}_r - R \ddot{\phi}_r \cos \theta_r = 0, \quad \ddot{Y}_f - R \ddot{\phi}_f \cos \theta_f = 0 \end{aligned} \quad (8)$$

There is no analytical solution of reduced equations of motion, in general situation.

Constrained mechanical system-deformed equations:

Deformed equations are obtained by the physical approach, based on Chetav-type constraint force. Deformed equations of motion are:

$$\begin{aligned} & J_f \ddot{\phi}_f + \mu_3 R \sin \theta_f - \mu_4 R \cos \theta_f + \mu_5 + \dots \\ & \dots \frac{\mu_6}{\cos \theta_r} \dots \\ & \dots \mu_7 \frac{R}{l_r + l_f} (\cos \theta_r \tan(\theta_r + \psi \cos \lambda) - \sin \theta_r) = 0 \\ & J_f \ddot{\phi}_r + \mu_1 R \sin \theta_r - \mu_2 R \cos \theta_r - \dots \\ & \dots \frac{\mu_5}{\cos(\theta_r + \psi \cos \lambda)} + \dots \\ & \dots \mu_6 - \mu_8 \frac{R}{l_r + l_f} (\cos \theta_f \tan(\theta_f + \psi \cos \lambda) - \sin \theta_f) = 0 \\ & 2l_f m_f (\ddot{\theta}_f - \ddot{X}_f) + \mu_7 = 0; \quad 2l_r m_r (\ddot{\theta}_r - \ddot{X}_r) = 0 \\ & m_r \ddot{X}_r - m_r l_r \ddot{\theta}_r + \mu_1 = 0; \quad m_r \ddot{Y}_r + \mu_2 = 0 \\ & m_f \ddot{X}_f - m_f l_f \ddot{\theta}_f + \mu_3 = 0; \quad m_f \ddot{Y}_f + \mu_4 = 0 \end{aligned} \quad (8)$$

4. Conclusion

In this study we studied a real practical system subjected to a true nonholonomic constraint condition-a bicycle model for first time. Moreover, the presented results formulation indicate the effectiveness of the geometrical theory of nonholonomic constraints for formulating of motion of concrete nonholonomic constraints systems with constraints based on the assumption of rolling without slipping. In perspective of this research, of the authors focuses on numerical solution of reduced equations of motion and detailed.

References

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