ANALYTICAL MODELLING OF FLUID FLOW THROUGH A DEFORMABLE TUBE

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Abstract

In this document, we analytically investigate a fluid flow through a deformable tube. The fluid is considered to be newtonian, incompressible and it moves along an elastic and isotropic cylindrical wall. The study provides a review of recent modelling aimed at understanding the effects of fluid parameters over the elastic wall tube behaviour. The unsteady fluid flow will be analysed following the asymptotic approach process using to a large Reynolds number and a small aspect radio. Moreover, according the small axisymmetric deformation, the wall is mathematically developed basing on the thin shell theory whose linear approach is applied. Lastly, the dynamic behaviour of the tube is represented and discussed.

Keywords: Analytically, study, fluid, deformable, tube.

1. Introduction

Studying a fluid flow through a deformable tube is due to its occurrence for its diversity practice in many industrial systems and its capability to generate a variety of instabilities as using a rigid wall [1]. This standing is reflected in biology [2], in micro-fluidic devices [3,4], in the renewable energies [5], and Recently in the field of transporting gaseous materials under pressure [6] and in engineering [7,8].

Although much numerical and experimental progress has been made during the past decades, studying interaction between structure and fluid analytically is not absolutely understood yet and remains to be discovered.

The present work focuses an analytical analysis of the fluid flow aspect and its effect on the wall tube behaviour. It is based on asymptotic approach followed by a numerical simulation. The small parameter 'ε' characterizing the aspect ratio of the tube governs the fluid asymptotic expansion. Moreover, based on linear approach of the thin shell theory, the treatment of the wall tube equations motion is developed by asymptotic process founded on geodesic curvature parameter.

2. Formulation of the problem

2.1 Fluid

In the presence of gravity force, we analyze an unsteady flow of an incompressible, viscous and Newtonian fluid. ρ and ν denote respectively the fluid density and the kinematic viscosity, L is the tube length, h is the thickness and Ro is the radius at rest. $R(z',t')$ is the variable radius (radius is a function of the longitudinal variable and time).
We assume that the tube behaves as a homogenous and linear elastic shell with ρ_T is the tube density (Figure 1).

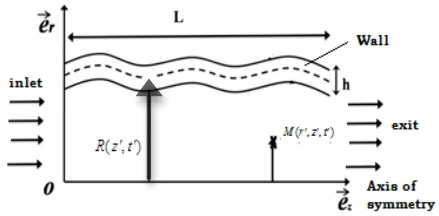


Figure 1. The deformed domain

The physical variables are denoted using primes. At this level, we introduce dimensionless variables, namely:

$$r=\frac{r'}{R_o}\ ,\ z=\frac{z'}{L}\ ,\ t=\frac{t'}{T_f},u=\frac{u'}{\varepsilon W_o}\ ,\ w=\frac{w'}{W_o}\ ,\ p=\frac{p'}{\rho\varepsilon^2 W_o^2} \tag{1}$$

T_f is the reference time, $\varepsilon=\frac{R_0}{L}\ll 1$ is the aspect ratio and W_o represents the inlet axial velocity.

Using system (1), the dimensionless Navier-Stokes and continuity equations of the problem, read as:

$$\begin{cases}(\frac{S_t}{\varepsilon})\frac{\partial u}{\partial t}+(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z})=-\frac{\partial p}{\partial r}+(\frac{R_e^{-1}}{\varepsilon})\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\varepsilon^2\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right\}\\ (\frac{S_t}{\varepsilon})\frac{\partial w}{\partial t}+(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z})=-\varepsilon^2\frac{\partial p}{\partial z}+(\frac{R_e^{-1}}{\varepsilon})\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r})+\varepsilon^2\frac{\partial^2 w}{\partial z^2}\right\}\\ \frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}+\frac{u}{r}=0\end{cases} \tag{2}$$

At *large Reynolds* number [$R_e=(R_0W_0)/\upsilon$] and *low Strouhal* number [$S_t=R_0/(W_0T_f)$], the system (2) is valid under the asymptotic restriction:

$$\frac{S_t}{\varepsilon}\equiv O(1) \qquad \frac{R_e^{-1}}{\varepsilon}\equiv O(1) \tag{3}$$

These interactions are the key to respectively analyze the coupling between the timescales and the nature of fluid with space scales

2.2 Tube

According to the thin shell theory [9], the flexible tube is modelled by using the non-linear Kirchhoff–Love geometrical asymptions [10,11,12]. The following system formulates the dimensionless variables and parameters:

$$\begin{cases} \varepsilon_0=\dfrac{h}{R_0} \quad \bar{R}=\dfrac{R}{R_0} \quad t=\dfrac{t'}{T_T} \quad \varepsilon_1=\dfrac{h}{L} \quad \gamma_f^0=\dfrac{\rho.W_0^2}{\lambda_1+\lambda_2} \quad \beta_2=\dfrac{\bar{u}_{0.2}}{L} \\ \bar{P}_3^*=\dfrac{P_3^*}{\rho.W_0^2\varepsilon^2}=\dfrac{P_{\text{int}-tube}-P_{\text{ext}-tube}\Big|_{r'=R(z',t')-\frac{h}{2}}}{\rho.W_0^2\varepsilon^2} \quad \gamma_c=\dfrac{\rho_c.L^2}{(\lambda_1+\lambda_2).T_T^2} \quad H=\dfrac{\lambda_2}{\lambda_1+\lambda_2} \end{cases} \tag{4}$$

Where T_T is the tube time reference, λ_1 and λ_2 are the Lame constants and $\bar{u}_{0.2}$ is the wall axial displacement reference.

In this section, we adopt the *"Least Degeneration Principle LDP"* approach to better modeling the behavior. The following system states our asymptotic constraints.

$$\begin{cases} \varepsilon^2 \equiv \varepsilon_0 \quad \beta_2 \equiv \dfrac{1}{\bar{R}}\dfrac{\partial \bar{R}}{\partial z} \quad \varepsilon_1^2 \equiv \beta_2 \end{cases} \tag{5}$$

Where $\dfrac{1}{\bar{R}}\dfrac{\partial \bar{R}}{\partial z}$ is the tube geodesic curvature along the $\vec{e}_z$.

According the system (4), the governing equations for the tube motion are stated as the under system:

$$\beta_2 H\frac{\partial \bar{u}_2^*}{\partial z}-\beta_2^2\bar{u}_2^*+\gamma_c\beta_2^{1/3}\bar{R}\frac{\partial^2\bar{R}}{\partial t^2}-\left(1-\frac{1}{\bar{R}}\right)-\gamma_f^0\bar{R}\bar{P}_3^*=0 \tag{6}$$

$\bar{u}_2^*$ is the dimensionless wall axial displacement.

3. Linearized problems and resolutions

Fluid: Let us linearize equations (2) about the particular solution at the *inlet of tube*. Denoting by ε the linearization parameter, the *LDP* provides us the following form:

$$u=\varepsilon^3\bar{u} \; ; \; w=1+\varepsilon^3\bar{w} \quad p=\tilde{P}_{amb}+\varepsilon^3\bar{p} \tag{7}$$

Where $\bar{u},\bar{w},\bar{p}$ are, respectively, the perturbed radial and axial velocities, and pressure:

$$\bar{u}=\bar{w}=\bar{p}=0 \quad for \quad t=r=z=0 \tag{8}$$

Inserting (4) into (2), we obtain at order ε^2 included, the non-degenerate equations, namely:

The 0th order terms

$$\begin{cases} \dfrac{\partial\bar{u}}{\partial t}+\dfrac{\partial\bar{u}}{\partial z}=-\dfrac{\partial\bar{p}}{\partial r}+\left\{\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\dfrac{\partial\bar{u}}{\partial r})-\dfrac{\bar{u}}{r^2}\right\} \\ \dfrac{\partial\bar{w}}{\partial t}+\dfrac{\partial\bar{w}}{\partial z}=\left\{\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\dfrac{\partial\bar{w}}{\partial r})\right\} \\ \dfrac{\partial\bar{u}}{\partial r}+\dfrac{\partial\bar{w}}{\partial z}+\dfrac{\bar{u}}{r}=0 \end{cases} \tag{9}$$

The 1st order terms

$$\begin{cases} \bar{u}\dfrac{\partial\bar{u}}{\partial r}+\bar{w}\dfrac{\partial\bar{u}}{\partial z}=0 \\ \bar{u}\dfrac{\partial\bar{w}}{\partial r}+\bar{w}\dfrac{\partial\bar{w}}{\partial z}=0 \end{cases} \tag{10}$$

The 2nd order terms

$$\begin{cases} \dfrac{\partial^2\bar{u}}{\partial z^2}=0 \\ \dfrac{\partial^2\bar{w}}{\partial z^2}-\dfrac{\partial\bar{p}}{\partial z}=0 \end{cases} \tag{11}$$

At this level, we look for the linearized solutions. So, the 1st order terms are neglected. Under this assumption, the *analytical solution of the pressure is obtained.*

$$\begin{cases} \bar{p}=-\dfrac{A_1}{2}.\sqrt{2}.\sqrt{-I.\omega}J_0\left(\dfrac{1}{2}.\sqrt{2}.\sqrt{-I.\omega}.r\right).z+A_2.\sqrt{-I.\omega}J_0\left(\sqrt{-I.\omega}.r\right) \\ A_1=\dfrac{I.\omega.A_2\left[I.\gamma_f.\omega\sqrt{-I.\omega}J_0\left(\sqrt{-I.\omega}\right)+J_1\left(\dfrac{1}{2}\sqrt{-I.\omega}\right)\right]}{J_1\left(\dfrac{1}{2}.\sqrt{2}.\sqrt{-I.\omega}\right)} \end{cases} \tag{12}$$

Where the J_0 and J_1 are the Bessel functions and ω is the dimensionless fluid frequency. A_1 and A_2 are the complex numbers with I is the imaginary unit.

Tube: To resolve the equation (6), we take up the linearization process around the initial equilibrium state. We introduce the linearized parameters ($\beta_2 \ll 1$):

$$\bar{P}_3^*=\beta_2^{1/3}\bar{P}_3^0 \quad \bar{R}=1+\beta_2^{1/3}\bar{R}_1 \tag{13}$$

Inserting (13) into (6), the approached solution is formulated at the 0th order of $\beta_2^{1/3}$ by:

$$\gamma_f^0\bar{P}_3^0\Big|_{r=1}+\bar{R}_1=0 \tag{14}$$

This relation analytically presents the linear correlate to fluid flow pressure to the wall deformation. This is in totally agreement with many numerical models [12,13].

4. Application and interpretations

In order to investigate the dynamical behaviours of a three-dimensional flexible tube due to fluid-structure interaction, the geometrical and numerical parameters of the simulation are listed in Table1.

Table 1: *Geometrical and numerical parameters.*

Fluid (*SAE 50W*):	
- Density	: 902 kg.m^{-3}
- Dynamic viscosity	: 0.86 Pa.s^{-1}
- Strouhal number (T_{ref}=0.05 s)	: 0. 025
Tube (Rubber):	
- Density	: 990 kg.m^{-3}
- Young's modulus	: 10^7 Pa.s
- Poisson's ratio	: 0.4
- Length	: 0.8 m
- Radius (At rest)	: 1.5 cm
- Thickness	: 2 mm

With the above parameters, the asymptotic restriction (3) and the asymptotic constraints (5) are perfectly verified $\left(S_t=0.025, R_e=188, \varepsilon=0.018\right)$. Moreover, the resolution provides a relationship between the fluid-structure characteristics and *dimensionless fluid frequency* (Figure 2).

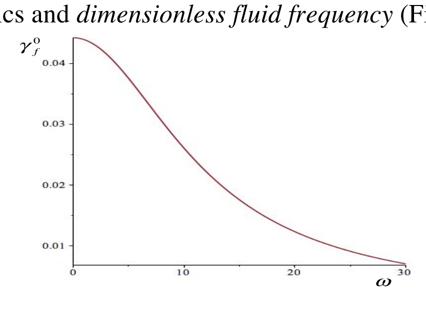
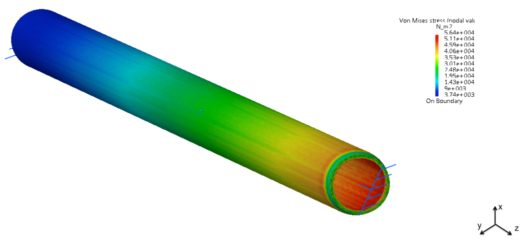
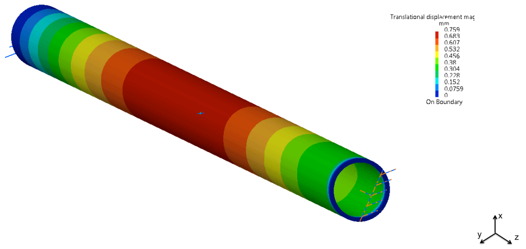


Figure 2. *System characteristics versus fluid frequency*



(a) Stress contour plot at t'=0.0125 s (t=0.25)



(b) Displacement of the tube wall at t'=0.0125 s (t=0.25)

Figure 3. *Three-dimensional tube behaviour at* F_{Fluid} *=20 Hz* ;

Figure (3) illustrates the elastic deformation which provides an assessment the degree of swelling of a rubber tube established by the fluid pressure. These results will be beneficial for a good control of the prevention widely used to preventing or minimizing the transmission of dynamic oscillations to a supporting structure [14]. In addition, we observe any stress overtaking of elastic limit and slight displacement of the wall.

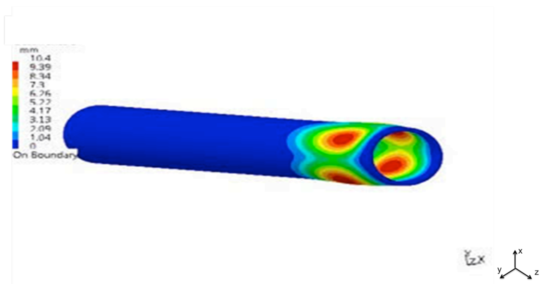


Figure 4. *Translational displacement in buckling analysis (Buckling factor = -37.361)*. at t'=0.0125 and F_{Fluid} *=20 Hz*

With this Buckling factor, the Figure.4 indicates the elastic pressure wave's propagation through the wall in the vicinity at the exit. The validity of solution is realized [15].

Conclusion

In this paper we represented analytically the solution of fluid structure interaction for a compressible fluid flowing through a deformable tube. Similarity solution is developed by Flaherty et al [16,17]. According to these findings, the dynamic behavior of a flexible tube used in industrial hydraulic systems has been performed.

As future work, the tube wall behavior will be processed considering the axial velocity with a large interval frequency. This process will allow us to determine the rate limit of wall radial displacement in order to validate the asymptotic approach of the model as small deformations.

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