Instabilities and bifurcations in thermal convection of viscoelastic fluids saturating a porous square box

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Abstract

We report theoretical and numerical results on bifurcations in thermal instability for a viscoelastic fluid saturating a porous square cavity heated from below. Temporal stability analysis showed that the first bifurcation from the conductive state may be either oscillatory for sufficiently elastic fluids or stationary for weakly elastic fluids. The dynamics associated with the nonlinear interaction between the two kinds of instabilities is first analyzed in the framework of a weakly nonlinear theory. On the other hand, computations performed with high Rayleigh number for weakly and strongly elastic fluids indicated that the system exhibits successive bifurcations from stationary or oscillatory single-cell convection to a more complex spatio-temporal multi-cellular flows. The major new findings were presented in the form of bifurcations diagrams as functions of viscoelastic parameters for Rayleigh number up to 420.

Keywords: **Porous media, thermal convection, viscoelastic fluids, nonlinear stability.**

1. Introduction

Viscoelastic fluids can be found in a great number of applications such as those in bio-engineering and in pharmaceutical and petroleum industries, among others. Recently, some activities have been devoted to investigate the primary convection patterns of a viscoelastic fluid confined in a porous medium heated from below by using the modified Darcy's law based on the Oldroyd-B model. Kim et al. [1] and Yoon et al. [2] performed a linear stability analysis and showed that in viscoelastic fluids such as polymeric liquids, a Hopf bifurcation as well as a stationary bifurcation may occur depending on the magnitude of the viscoelastic parameters. With infinite horizontal porous cavity, the question of whether standing or traveling waves are preferred at onset has been fully addressed by Hirata et al [3]. In addition to its theoretical interest, Delenda et al [4] have showed that viscoelastic convection in porous media may be useful for industrial applications interested by the separation of species of viscoelastic solutions. The objective of this study is to use both theoretical and numerical approaches in order determine a global picture in the Rayleigh-viscoelastic parameters space on possible successive bifurcations of convection patterns in a square porous cavity saturated by a viscoelastic fluid.

2. Problem formulation and review of linear stability

We consider a square box filled with a Boussinesq viscoelastic liquid saturating a porous square cavity heated from below with a constant temperature. Horizontal boundaries are assumed perfectly heat conducting while the vertical walls are considered impermeable and adiabatic. The dimensionless form of the governing equations are [5]

$$\begin{cases} \partial_t T = (\partial_x^2 + \partial_z^2 - u\partial_x - w\partial_z)T \\ (1+\lambda_1\partial_t)\mathcal{H} = \mathfrak{R}(1+\lambda_2\partial_t)\partial_x T \\ (\partial_x^2+\partial_z^2)\psi = \mathcal{H} \\ T(x,0) = 1; \quad T(x,1) = \partial_x T(0,z) = \partial_x T(1,z) = 0 \\ \psi(x,z) = 0 \quad \textit{for all boundaries} \end{cases}$$

where λ_1 and λ_2 are the relaxation time and the retardation time respectively, $\mathfrak{R} = g\beta\Delta T \mathrm{KH}/(\nu\kappa)$ is the Darcy-Rayleigh number and ψ is the stream function .

We shall briefly sketch the linear analysis, by emphasizing the most important points. When the conductive basic state loses its stability, There is a line of steady bifurcations given by $\mathfrak{R}_c^s = 4\pi^2$ for a single-cell convection and a line of Hopf bifurcations along:

$$\mathfrak{R}_c^{osc}(m,\lambda_1,\lambda_2) = \frac{1+m^2}{\lambda_1 m^2}\left(1+\lambda_2\pi^2(1+m^2)\right) \tag{11}$$

with the critical frequency:

$$\omega_c(m) = \sqrt{\frac{1}{\lambda_2}[\pi^2(1+m^2)(1-\frac{\lambda_2}{\lambda_1}) - \frac{1}{\lambda_1}}$$

provided $\lambda_1 > \lambda_1^\star = \lambda_2 + \frac{1}{\pi^2(1+m^2)}$,

where m is the number of rolls. The Hopf curve branches off the steady bifurcation curve when

$$\mathfrak{R} = \mathfrak{R}^* = \mathfrak{R}_c^{osc} = 4\pi^2, \lambda_1 = \lambda_1^\star = \lambda_2 + \frac{1}{\pi^2(1+m^2)}$$

At this point the linear theory fails to predict the dominant mode of convection. Therefore, a weakly nonlinear stability analysis is needed to elucidate the bifurcation processes near the codimension-two bifurcation point $(\mathfrak{R}^*, \lambda_1^\star)$.

3. **Weaky nonlinear stability results**

In a recent paper Hirata et al. [3] proposed a nonlinear reduced model by using perturbation techniques in the neighborhood of the codimension-two bifurcation point $(\mathfrak{R}^*, \lambda_1^\star)$. They derived the following system of nonlinear ordinary differential equations governing the dynamics of the normalized vertical velocity field W:

$$\begin{cases} \dfrac{dW}{dt} = Z(t) & (16.a) \\ \dfrac{dZ}{dt} = -W^3 + \mu_1 W + \mu_2 Z(t) - W^2 Z(t) & (16.b) \end{cases}$$

with $\mu_1 = \frac{f_2{}^2}{f_1{}^2}\frac{\mathfrak{R}-\mathfrak{R}_c^s}{2\lambda_2}$, $\mu_2 = \frac{f_2}{f_1}\frac{\lambda_1^\star(\mathfrak{R}-\mathfrak{R}_c^{osc})}{2\lambda_2}$, $f_1 = \frac{1}{8\lambda_2}$

and $f_2 = -\frac{1}{\lambda_2}[\frac{5}{16\pi^2} - \frac{3\lambda_1^\star}{8} + \frac{3}{8\lambda_2}(\frac{1}{4\pi^4} - \frac{\lambda_1^\star}{2\pi^2})]$

The stability types of fixed points and limit cycles, the bifurcation lines and the phase portrait associated with the above system were analyzed by Hirata et al. [3] by using dynamic systems theory. In particular, we succeeded in obtaining an explicit form of some nonlinear thresholds as a function of viscoelastic parameters :

$$\mathfrak{R}_{NL}(\alpha) = 4\pi^2\left(1 + \frac{\omega_c^2\lambda_1\lambda_2^\star}{\alpha - 1 + 2\pi^2\,\lambda_2(3\alpha-1)}\right) \tag{17}$$

with **:**

$\mathfrak{R}_{NL1} = \mathfrak{R}_{NL}(\alpha=1)$, $\mathfrak{R}_{NL2} = \mathfrak{R}_{NL}(\alpha=4/5)$ and $\mathfrak{R}_{NL3} = \mathfrak{R}_{NL}(\alpha=0.752)$

. The oscillatory convection induced by the Hopf bifurcation is the only stable pattern for $\mathfrak{R} < \mathfrak{R}_{NL1}$. The two types of convective patterns, namely oscillatory and stationary convection coexist between the nonlinear thresholds $\mathfrak{R}_{NL1}$ and $\mathfrak{R}_{NL3}$. The observability of either oscillatory convection or stationary one depends on the initial conditions. The line $\mathfrak{R} = \mathfrak{R}_{NL2}$ corresponds to a double homoclinic bifurcation points. The line $\mathfrak{R} = \mathfrak{R}_{NL3}$ represents the nonlinear threshold for the transition from oscillatory convection to stable stationary one independently of initial conditions. This means physically that the viscoelasticity of the fluid has no influence on the convection properties and the system behaves like a Newtonian fluid when the Rayleigh number slightly exceeds the nonlinear threshold $\mathfrak{R}_{NL3}$.

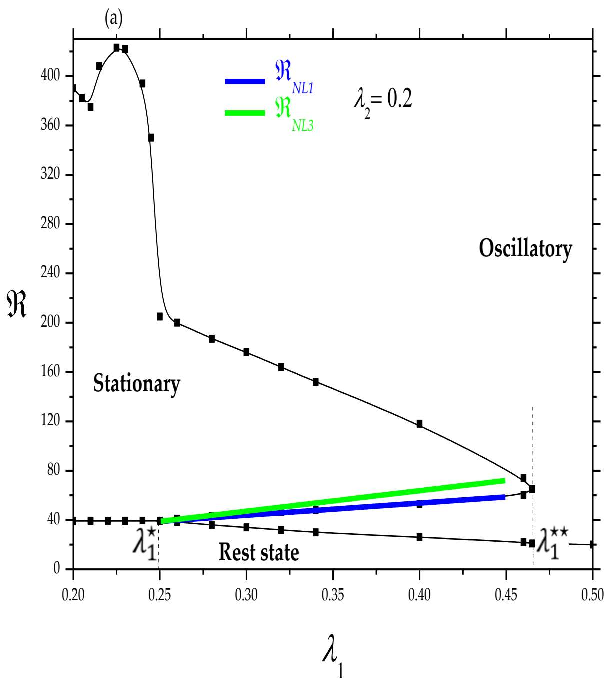
4. Numerical results

One of our main points of interest centers on the question as to whether the primary and the secondary steady bifurcation observed respectively in the weakly and moderately viscoelastic regimes is stable against time- dependent disturbances. With regard to this question, it has been established that for a Newtonian fluid saturating a porous square box, a single-cell solution undergoes a series of bifurcations as the Rayleigh number is increased. At the second critical value of the Rayleigh number $\mathfrak{R} \approx 390$, a Hopf bifurcation has been observed for two-dimensional single-cell convection [6] .

The general bifurcation diagram is shown in $(\lambda_1, \mathfrak{R})$ plane for $\lambda_2 = 0.2$ (Fig. 1(a)), and in $(\lambda_2, \mathfrak{R})$ plane for $\lambda_1 = 0.5$ (Fig. 1(b)) where the upper curves correspond to the computed Hopf bifurcation lines up to $\mathfrak{R} = 420$. In order to have an overview summarizing in a single scheme all the main bifurcations, we also display in both figures the transition line to a first Hopf bifurcation determined by linear stability analysis (lower curves) and the transition line to a steady convection state (middle curves) determined previously. The two figures present a few interesting phenomena worth discussing in more detail. First among them concerns the successive bifurcations observed in the weakly elastic fluids (i.e. for $\lambda_1 < \lambda_1^\star = 0.25$ in Fig. 1(a) or $\lambda_2 > \lambda_2^\star = 0.45$ in Fig. 1(b). For these fluids, a first stationary bifurcation occurs at the well known critical Rayleigh number $\mathfrak{R}_c^s = 4\pi^2$. This means that the fluid elasticity has no effect on the first instability properties and the non-Newtonian fluid behaves as a Newtonian fluid. By increasing $\mathfrak{R}$ to a second critical value $\mathfrak{R}_2^{Osc}$, a Hopf bifurcation occurs and the steady convective pattern is replaced by oscillatory convection. A close inspection of Fig. 1(a) and Fig. 1(b) shows that the second critical Rayleigh number $\mathfrak{R}_2^{Osc}$ reaches its maximum for $\lambda_1 = \lambda_1^\star = 0.25$ and $\lambda_2 = \lambda_2^\star = 0.45$ respectively. When λ_1 is decreased from $\lambda_1^\star$ to $\lambda_1 = \lambda_2 = 0.2$ (see Fig. 1(a)) or λ_2 is increased from $\lambda_2^\star$ to $\lambda_2 = \lambda_1 = 0.5$ (see Fig. 1(b), $\mathfrak{R}_2^{Osc}$ deceases from its maximum and eventually joins the limit $\mathfrak{R}_2^{Osc} = 390$, corresponding to Newtonian fluids [6]. We conclude that the effect of the relaxation (retardation) time on the second transition to oscillatory convection is stabilizing (destabilizing) for weakly viscoelastic fluids.

Second, in moderately viscoelastic regime where the viscoelastic parameters are such that $\lambda_1 \in]\lambda_1^\star, \lambda_1^{\star\star}[$ or $\lambda_2 \in]\lambda_2^{\star\star}, \lambda_2^\star[$, three sources of instability act in a concert to select the dominant mode of instability, namely the heating from below, the relaxation time and the retardation time. The resulting dynamics in this regime may be understood for example by increasing Darcy-Rayleigh number for fixed values of viscoelastic parameters. The first convective instability is oscillatory rather than steady when Darcy-Rayleigh number exceeds the onset of the first

Hopf bifurcation $\Re_c^{Osc}$ (region between the lower line and the middle line in Fig. 1(a) and in Fig. 1(b)). It should be noted that this oscillatory convection is completely due to the viscoelastic character of the fluid. By increasing $\Re$ to a defined critical value, a secondary bifurcation occurs where a stationary pattern becomes the dominant mode of instability (region between the middle line and the upper line in Fig. 1(a) and in Fig. 1(b)). Physically, this transition may be understood by the dominant viscous effect compared to the elastic contribution. Finally, at a third critical Rayleigh number $\Re_3^{Osc}$, a transition is observed as a secondary Hopf bifurcation giving rise to a new mode of oscillatory convection (region beyond the upper lines in Fig. 1(a) and in Fig. 1(b)). It is interesting to note that the critical Rayleigh number $\Re_3^{Osc}$ observes a sharp decrease (i.e. a strongly destabilizing effect) when λ_1 exceeds the value $\lambda_1^\star$ by small amount or λ_2 is just below the value $\lambda_2^\star$.



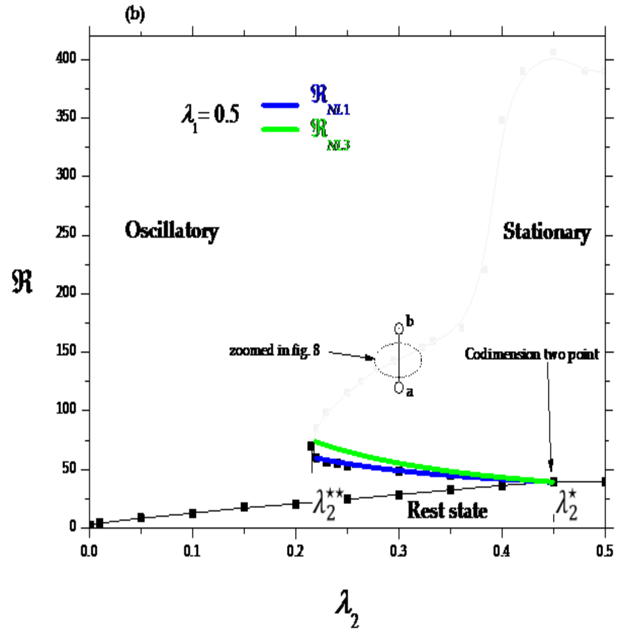


Figure 1: General stability diagram for $\lambda_2 = 0.2$ in the $(\lambda_1, \Re)$ plane (a) and for $\lambda_1 = 0.5$ in the $(\lambda_2, \Re)$ plane (b). $\Re_{NL1}$ (the lower bold line) and $\Re_{NL3}$ (the upper bold line) correspond to the nonlinear thresholds derived from weakly nonlinear stability analysis.

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